Example Sheet #2

(14) Work over the alphabet $\Sigma = \{0, 1\}$ and for $n \geq 1$ consider

$$L_n := \{w\,;\text{ there are } x, y \in \mathbb{W} \text{ such that } |y| = n-1 \text{ and } w = x1y)\},$$

i.e., the set of words that have a 1 in the nth position counted from the end of the word. Show that:

(a) There is a nondeterministic automaton N with $n+1$ states such that $\mathcal{L}(N) = L_n$ and

(b) if D is a deterministic automaton with fewer than 2^n states, then $\mathcal{L}(D) \neq L_n$.

(15) Let R, S, T be regular expressions. For each of the following statements, either prove that it is true, or find a specific counterexample.

(a) $\mathcal{L}(R(S+T)) = \mathcal{L}(RS+RT)$;

(b) $\mathcal{L}((R+S)T) = \mathcal{L}(RT+ST)$;

(c) $\mathcal{L}(R+ST) = \mathcal{L}(RS+RT)$;

(d) $\mathcal{L}(R^*R) = \mathcal{L}(R^+)$;

(e) $\mathcal{L}((R+SR^*)^*) = \mathcal{L}((R+S)^*)$;

(f) $\mathcal{L}((R+S)^+) = \mathcal{L}(R^+ + S^+)$

(16) Let L and M be languages over an alphabet Σ and consider the set equation $X = LX \cup M$. Prove the following statements:

(a) The language L^*M is a solution for this equation.

(b) If Z is a solution for this equation, then $L^*M \subseteq Z$.

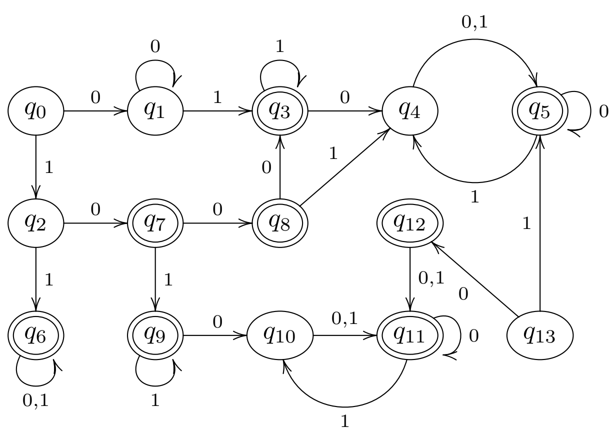
(c) If $\varepsilon \notin L$, then L^*M is the only solution for this equation.

(17) Give regular expressions that describe the languages given by the grammars $G = (\Sigma, V, P, S)$

(a) where $P = \{S \to aS, S \to a\}$ and

(b) where $P = \{S \to cS, S \to cA, A \to aA, A \to aB, B \to bB, B \to b\}$.

(18) Minimise the following automaton using the construction given in the lectures.



(19) Give the minimal deterministic automaton for the language L_n discussed in (14).

(20) Let G be the context-free grammar given by $S \to ABS, S \to AB, A \to aA, A \to a, B \to bA$. For each of the words *aabaab*, *aaaaba*, *aabbaa*, *abaaba*, either draw the parse tree that shows that the word lies in $\mathcal{L}(G)$ or argue that it does not lie in $\mathcal{L}(G)$.

(21) Consider $\Sigma = \{0, 1, \varepsilon, (,), +, \varnothing, ^+, ^*\}$. Define a context-free grammar G over Σ such that $\mathcal{L}(G)$ is the set of regular expressions over the alphabet $\{0, 1\}$. Show that you cannot find a regular grammar with this property.

(22) If $v \in \mathbb{W}$, we call u a *subword* of v if there are $x, y \in \mathbb{W}$ such that $v = xuy$. In that case, we call the word xy the *result of removing the subword u from v*. For $e, w \in \mathbb{W}$, we say that e is an *excerpt* of w if it is the result of removing finitely many subwords from w, i.e., $w = x_0y_0x_1y_1 \dots x_ny_nx_{n+1}$ and $e = x_0 \dots x_{n+1}$ for some $x_0, \dots, x_{n+1}, y_0, \dots, y_n \in \mathbb{W}$; we say that it is a *proper excerpt* if $e \neq w$.

(a) Suppose that $L := \mathcal{L}(D)$ for a deterministic automaton D with $|Q| = n$ and that $w = xvy \in L$ for $x, v, y \in \mathbb{W}$ with $|v| \geq n$. Prove that there is a proper excerpt e of v such that $xey \in L$.

(b) Show that the following language is context-free, not regular, but satisfies the regular pumping lemma with pumping number $n = 2$:

$$\Sigma = \{0, 1, 2\}, \quad L = \{w20^n1^n\, ;\, w \in \mathbb{W}, n > 0\} \cup \{0, 1\}^+.$$

[*Hint.* For non-regularity, use (a).]

(23) Convert the context-free grammar given by the following production rules to Chomsky normal form, explaining the transformation steps in detail: $S \to aSbb, S \to T, T \to bTaa, T \to S$.

(24) Give a context-free grammar in Chomsky normal form for the languages $\{a^mb^{2m}c^k\, ;\, m, k \geq 1\}$ and $\{a^mb^ka^m\, ;\, m, k \geq 1\}$. Justify your answer.

(25) Let $G = (\Sigma, V, P, S)$ be a context-free grammar in Chomsky normal form. Let

$$P' := P \cup \{A \to \varepsilon\, ;\, \text{there is some } a \in \Sigma \text{ such that } A \to a \in P)\}$$

and $G' := (\Sigma, V, P', S)$. Describe the language $\mathcal{L}(G')$ in words, giving an argument for your answer. What changes if G is not in Chomsky normal form?

(26) Suppose $\Sigma = \{a\}$ and that L is a language over Σ that satisfies the context free pumping lemma. Show that it satisfies the regular pumping lemma.

(27) For each of the following languages, decide whether it is context-free and provide an argument for your claim:

(a) $\{a^nb^m\, ;\, n \neq m, n + m > 0\}$;
(b) $\{a^mb^nc^md^n\, ;\, m, n \geq 1\}$;
(c) $\{a^nb^mc^kd^\ell\, ;\, 2n = 3m \text{ and } 5k = 7\ell\}$;
(d) $\{a^nb^mc^kd^\ell\, ;\, 2n = 3k \text{ and } 5m = 7\ell\}$;
(e) $\{a^nb^mc^kd^\ell\, ;\, 2n = 3k \text{ or } 5m = 7\ell\}$;
(f) $\{a^p\, ;\, p \text{ is prime}\}$;
(g) $\{a^{2^n}\, ;\, n \geq 1\}$;
(h) $\{ww\, ;\, w \in \{a, b\}^+\}$;
(i) $\{a, b\}^+ \backslash \{ww\, ;\, w \in \{a, b\}^+\}$.

(28) Show that the class of context-free grammars is closed under the Kleene plus operation, i.e., if L is context-free, then so is L^+.