Example Sheet 4

Renewal theory and applications.

1. A cinema in Camford shows one movie at a time, which is changed after a time uniformly distributed in $[0, 3]$ (months). Purchasing the rights to a new movie costs £10,000. How much have they spent after 5 years?

Suppose that a given customer goes to the movies at rate once every two months, but only walks into the cinema if the movie is less than one month old. In 5 years, how many movies has she seen? What if the time to change a movie is exponentially distributed with mean 1.5 months?

2. Suppose that the lifetime of a car is a random variable with density function f. Mr. Smith buys a new car as soon as the old one breaks down or reaches T years. A new car costs $\pounds c$, while an additional $\pounds a$ is incurred if the car breaks down before T. In the long-run, how much does Mr. Smith spend on his cars per unit time?

3. Let $(X_t)_{t\geq 0}$ be a renewal process with interarrival times having the Gamma $(2, \lambda)$ distribution. Determine the limiting excess distribution. Determine also the expected number of renewals up to time t.

4. A barber takes an exponentially distributed amount of time, with mean 20 minutes, to complete a haircut. Customers arrive at rate 2 per hour, but leave if both chairs in the waiting room are full. Make a Markov chain model on the state space $\{0, 1, 2, 3\}$. Then using Little's formula, find the average waiting time in the system (including his service time) of a customer.

Spatial Poisson processes.

5. In a certain town at time $t = 0$ there are no bears. Brown bears and grizzly bears arrive as independent Poisson processes B and G with respective intensities β and γ.

(a) Show that the first bear is brown with probability $\beta/(\beta + \gamma)$.

(b) Find the probability that between two consecutive brown bears, there arrive exactly r grizzly bears.

(c) Given that $B(1) = 1$, find the expected value of the time at which the first bear arrived.

6. **Campbell–Hardy theorem.** Let Π be the points of a non-homogeneous Poisson process on $\mathbb{R}^d$ with intensity function λ. Let $S = \sum_{x\in\Pi} g(x)$ where g is a smooth function which we assume for convenience to be non-negative. Show that $\mathbb{E}(S) = \int_{\mathbb{R}^d} g(u)\lambda(u)\,du$ and $\mathrm{var}(S) = \int_{\mathbb{R}^d} g(u)^2\lambda(u)\,du$, provided these integrals converge.

7. Let Π be a Poisson process with constant intensity λ on the surface of the sphere of $\mathbb{R}^3$ with radius 1. Let P be the process given by the (X, Y) coordinates of the points projected on a plane passing through the centre of the sphere. Show that P is a Poisson process, and find its intensity function.

8. Repeat the previous exercise, when Π is a homogeneous Poisson process on the ball $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq 1\}$.