Example Sheet 2

1. For any $r \times r$ matrix A, show that the exponential e^A is a finite dimensional matrix and if A_1 and A_2 commute, then

$$e^{A_1+A_2} = e^{A_1} \cdot e^{A_2}\,.$$

Calculate $P(t) = e^{tQ}$ where

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

2. Let $(X_t)_{t\geq 0}$ be a Markov chain on the integers with transition rates $q_{i,i+1} = \lambda q_i$, $q_{i,i-1} = \mu q_i$, and $q_{i,j} = 0$ if $|j-i| \geq 2$, where $\lambda + \mu = 1$ and $q_i > 0$ for all i. Write down the Q-matrix of this chain and draw the diagram corresponding to this chain. Then

(a) Find the probability, starting from 0, that X_t hits i, for any $i \geq 1$.
(b) Compute the expected total time spent in state i, starting from 0.

3. A salesman flies around between Amsterdam, Berlin, and Cambridge as follows. She stays in each city for an exponential amount of time with mean 1/2 month if the city is B or C, but with mean 1/4 month if the city is A. From A she goes to B or C with probability 1/2 each; from B she goes to A with probability 3/4 and to C with probability 1/4; from C she always goes back to A. (a) Find the limiting fraction of time she spends in each city. (b) What is her average number of trips each year from Berlin to Amsterdam?

4. Consider a fleet of N buses. Each bus breaks down independently at rate μ, when it is sent to the depot for repair. The repair shop can only repair one bus at a time and each bus takes an exponential time of parameter λ to repair. Find the equilibrium distribution of the number of buses in service.

5. N frogs are playing near a pond. When they are in the sun they are too hot so jump in the pond at rate 1. When they are in the pond they are too cold so jump out at rate 2. Write down the Q-matrix for X_t, the number of frogs in the sun at time t. Show that $\pi_i = \binom{N}{i} p^i (1-p)^{N-i}$ is an invariant distribution for this chain, for some suitable value of $p \in (0,1)$. Explain.

6. Customers arrive at a certain queue in a Poisson process of rate λ. The single 'server' has two states A and B, state A signifying that he is 'in attendance' and state B that he is having a tea-break. Independently of how many customers are in the queue, he fluctuates between these states as a Markov chain Y on $\{A, B\}$ with Q-matrix

$$\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

The total service time of any customer is exponentially distributed with parameter μ and is independent of the chain Y and of the service times of other customers.

Show that there exists $\theta \in (0,1]$, such that for all $k \geq 0$

$$\mathbb{P}(X \text{ hits } A_0 | X_0 = A_k) = \theta^k$$

where A_k denotes the state of the chain where there are k customers and the server is in state A.

Show that $(\theta - 1)f(\theta) = 0$, where

$$f(\theta) = \lambda^2\theta^2 - \lambda(\lambda + \mu + \alpha + \beta)\theta + (\lambda + \beta)\mu\,.$$

By considering $f(1)$ or otherwise, prove that X is transient if $\mu\beta < \lambda(\alpha + \beta)$, and explain why this is intuitively obvious.

7. Consider the continuous-time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ with non-zero transition rates

$$q_{i,i-1} = i^2 + 1, \quad q_{ii} = -2(i^2 + 1), \quad q_{i,i+1} = i^2 + 1.$$

Is $(X_t)_{t\geq 0}$ recurrent? Is $(X_t)_{t\geq 0}$ positive recurrent?

8. Consider the continuous-time Markov chain $(Y_t)_{t\geq 0}$ on $\mathbb{Z}$ with non-zero transition rates

$$q_{i,i-1} = 3^{|i|}, \quad q_{ii} = -3^{|i|+1}, \quad q_{i,i+1} = 2 \cdot 3^{|i|}.$$

Show that Y is transient. Show that Y has an invariant distribution. Explain.

9. Calls arrive at a telephone exchange as a Poisson process of rate λ, and the lengths of calls are independent exponential random variables of parameter μ. Assuming that infinitely many telephone lines are available, set up a Markov chain model for this process.

Show that for large t the distribution of the number of lines in use at time t is approximately Poisson with mean λ/μ.

Show that the expected number of lines in use at time t, given that n are in use at time 0, is $ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$

10. Let $G = (V, E)$ be a finite weighted graph: so each edge $e = (x, y)$ is endowed with a weight $w_{xy} \geq 0$. We assume that $w_{xy} = w_{yx}$, so the weights are undirected. Consider the Simple Random Walk $(X_t, t \geq 0)$ on G, which is the Markov chain on V with transition rates $q_{xy} = w_{xy}$ for $x \neq y$.

Pick a vertex $v \in V$ uniformly at random, and given v, consider two independent simple random walks $(X_s, s \geq 0)$ and $(X'_s, s \geq 0)$ started from v. Show that $d(X_t, X'_t)$ has the same distribution as $d(v, X_{2t})$, where $d(x, y)$ denote the graph distance between x and y: the smallest number of edges separating x from y.

11. Let Q be irreducible and let $(X_t, t \geq 0)$ be the corresponding continuous-time Markov chain. Fix $h > 0$ and let $Z_n = X_{nh}, n \geq 0$. Show that Z is a discrete-time Markov chain and give its transition matrix. Show that X is recurrent if and only if Z is recurrent.

Show that X is positive recurrent if and only if Z is positive recurrent in the non-explosive case. Do they have the same invariant distribution?

12. Let X be a non-explosive irreducible Markov chain on S and let π be a reversible invariant distribution. Let $E = \{f : S \to \mathbb{R} : \sum_x f^2(x)\pi(x) < \infty\}$ and define an inner product on E by setting

$$(f|g)_\pi = \sum_x f(x)g(x)\pi(x).$$

(a) Let $q_t(x, y) = p_t(x, y)/\pi_y$ and let q_t^x be the function defined by $q_t^x(y) = q_t(x, y)$. Show that $q_t(x, y) = q_t(y, x)$. Show that

$$q_{t+s}(x, y) = (q_t^x | q_s^y)_\pi$$

and deduce that $q_t^x \in E$.

(b) Define a quadratic form

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x,y} \pi_x q_{xy} (f(y) - f(x))(g(y) - g(x)),$$

whenever this is defined. The quantity $\mathcal{E}(f, g)$ is sometimes called the Dirichlet form of the Markov chain, and $\mathcal{E}(f, f)$ is called the Dirichlet energy of the function f. Show that

$$(Qf|g)_\pi = -\mathcal{E}(f, g).$$

(In particular this shows Q is self-adjoint).

(c) Fix $x \in S$ and let $\psi(t) = q_{2t}(x, x)$. Deduce that ψ is decreasing and then convex.