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Example Sheet 1

1. Arrivals of the Number 1 bus form a Poisson process of rate 1 bus per hour, and arrivals of the Number 7 bus form an independent Poisson process of rate 7 buses per hour.

(a) What is the probability that exactly 5 buses pass by in 1 hour?

(b) What is the probability that exactly 3 Number 7 buses pass by while I am waiting for a Number 1?

(c) When the maintenance depot goes on strike half the buses break down before they reach my stop. What then is the probability that I wait for 30 minutes without seeing a single bus?

2. Let $T_1, T_2, \dots$ be independent exponential random variables of parameter λ. Show that, for all $n \geq 1$, the sum $S = \sum_{i=1}^{n} T_i$ has the probability density function

$$f_S(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, x > 0.$$

This is called the Gamma (n, λ) distribution.

Let N be an independent geometric random variable with

$$\mathbb{P}(N = n) = \beta(1-\beta)^{n-1}, \qquad n = 1, 2, \dots \,.$$

Show that $T = \sum_{i=1}^{N} T_i$ has exponential distribution of parameter $\lambda\beta$. Deduce another proof of the thinning property of Poisson processes.

3. Customers arrive in a supermarket as a Poisson-process of rate N. There are N aisles in the supermarket and each customer selects one of them at random, independently of the other customers. Let X_t^N denote the proportion of aisles which remain empty at time t and let T^N denote the time until half the aisles are busy (have at least one customer). Show that

$$X_t^N \to e^{-t} \quad , \quad T^N \to \log 2$$

in probability as $N \to \infty$.

4. A pedestrian wishes to cross a single lane of fast-moving traffic. Suppose the number of vehicles that have passed by time t is a Poisson process of rate λ, and suppose it takes time a to walk across the lane. Assuming the pedestrian can foresee correctly the times at which vehicles will pass by, how long on average does it take to cross over safely?

Hint: *Let T be the time to cross and J_1 the time at which the first car passes. Identify the contributions to $\mathbb{E}\,T$ from the events $\{J_1 > a\}$ and $\{J_1 < a\}$.*

How long on average does it take to cross two similar lanes (a) when one must walk straight across, (b) when an island in the middle of the road makes it safe to stop half way? (Assume the traffic is independent in each direction and has same rate λ).

5. Customers enter a supermarket as a Poisson process of rate 2. There are two salesmen near the door who offer passing customers samples of a new product. Each customer takes an exponential time of parameter 1 to think about the new product, and during this time occupies the full attention of one salesman. Having tried the product, customers proceed into the store and leave by another door. When both salesmen are occupied, customers walk straight in. Assuming that both salesmen are free at time 0, find the probability that both are busy at a later time t.

6. (a) Let $(N_t, t \geq 0)$ be a Poisson process with rate $\lambda > 0$ and let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s,x)$ is a function and T_j are the jump times of N then

$$\mathbb{E}[\exp\{\theta \sum_{j=1}^{N_t} g(T_j, X_j)\}] = \exp\{\lambda \int_0^t (\mathbb{E}(e^{\theta g(s,X)}) - 1) ds\}.$$

This is called Campbell's theorem.

(b) Cars arrive at the beginning of a long road in a Poisson stream of rate λ from time $t = 0$ onwards. A car has a fixed velocity V miles per hour, where $V > 0$ is a random variable. The velocities of cars are independent and identically distributed, and independent of the arrival process. Cars can overtake each other freely. Show that the number of cars on the first x miles of the road at time t has a Poisson distribution with mean $\lambda \mathbb{E}[\min\{t, x/V\}]$.

7. A bi-infinite Poisson process $(N_t : t \in \mathbb{R})$ with $N_0 = 0$ is a process with independent and stationary increments over $\mathbb{R}$. Moreover, for all $-\infty < s \leq t < \infty$, the increment $N_t - N_s$ has the Poisson distribution with parameter $\lambda(t-s)$. Prove that such a process exists.

8. For a simple birth process (X_t) of parameter λ starting with one individual, find $\mathbb{E}(X_t)$.

9. Each bacterium in a colony splits into two identical bacteria after an exponential time of parameter λ, which then split in the same way but independently. Let X_t denote the size of the colony at time t, and suppose $X_0 = 1$. Show that the probability generating function $\phi(t) = \mathbb{E}(z^{X_t})$ satisfies

$$\phi(t) = ze^{-\lambda t} + \int_0^t \lambda e^{-\lambda s} \phi(t-s)^2 ds$$

and deduce that, for $q = 1 - e^{-\lambda t}$, and $n = 1, 2, \ldots$,

$$\mathbb{P}(X_t = n) = q^{n-1}(1-q)\,.$$

10. Let $(q_k)_{k\geq 1}$ be a sequence of positive numbers such that $q = \sum_k q_k < \infty$. Let $(E_k)_{k\geq 1}$ be a sequence of independent exponential random variables where the rate (i.e., parameter) of E_k is q_k. Let $E = \inf\{E_k\}$ and let $K = \arg\min\{E_k\}$, with $K = \infty$ if the inf is not attained. Compute $\mathbb{P}(K = k, E > t)$ for all $t > 0$ and $k \geq 1$, and hence identify the joint distribution of K and E.

Let Q be a Q-matrix on a countable state space S. After recalling the first two constructions of a Markov chain based on the holding times and jump chain, deduce from the above that the two constructions are equivalent.

11. Let Q be a Q-matrix on a finite state space S, and let $f(t) = \det P(t)$ where $P(t)$ is the associated transition matrix. Show that $f(t+s) = f(t)f(s)$. Deduce that $f(t)$ is of the form e^{tq}, and identify q by taking $t \to 0$.

12. Let S be finite and let $f : S \to \mathbb{R}$ be a function, identified with the vector $(f(x))_{x\in S}$. Let Q be a Q-matrix. Show that

$$Qf(x) = \sum_y q_{xy}(f(y) - f(x)),$$

where Qf is the standard matrix multiplication.

Let $\phi(t) = \mathbb{E}_x(f(X_t))$, where $(X_t, t \geq 0)$ is the Markov chain with Q-matrix given by Q. Deduce that

$$Qf(x) = \phi'(0) = \lim_{t\to 0} \frac{\mathbb{E}_x(f(X_t)) - f(x)}{t}.$$

Finally, deduce that

$$\mathbb{E}_x(f(X_t)) = f(x) + \int_0^t \mathbb{E}_x(Qf(X_s))ds.$$