Lent 2026 **Analysis of Functions – Example Sheet 3** P.P. Varjú

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A supervisor sheet will be available on Moodle or by email.

1. Recall that $L^\infty(\mathbf{R}) = L^1(\mathbf{R})'$. Consider the sequence $(f_n)_{n=1}^\infty$, where $f_n \in L^\infty(\mathbf{R})$ is given by $f_n(x) = \sin(nx)$. Show that $f_n \overset{*}{\rightharpoonup} 0$. Show that $f_n^2 \overset{*}{\rightharpoonup} g$ for some $g \in L^\infty(\mathbf{R})$ which you should find.

2. Suppose $f \in L^1(\mathbf{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

(1) Show that $\widehat{f} \in C^\infty(\mathbf{R}^n)$ and for any multi-index:

$$\sup_{\xi \in \mathbf{R}^n} |D^\alpha \widehat{f}(\xi)| \leq (2\pi R)^{|\alpha|} \|f\|_1.$$

(2*) This part requires some knowledge of complex analysis. Let $n = 1$ and show that $\widehat{f}$ extends to a holomorphic function on the complex plane. Conclude that it is not possible for both a function and its Fourier transform to be compactly supported.

3.

(1) Suppose $f \in \mathcal{S}(\mathbf{R})$. By observing that

$$\|f\|_2^2 = \int_{\mathbf{R}} \left(\frac{d}{dx} x\right) |f(x)|^2 dx,$$

or otherwise, show that:

$$\|f\|_2^2 \leq 4\pi \big\| |x| f(x) \big\|_2 \big\| |\xi| \widehat{f}(\xi) \big\|_2$$

with equality if and only if $f(x) = ae^{-\lambda |x|^2}$ for some $a \in \mathbf{C}, \lambda > 0$. Deduce that if $x_0, \xi_0 \in \mathbf{R}$:

$$\|f\|_2^2 \leq 4\pi \big\| |x - x_0| f(x) \big\|_2 \big\| |\xi - \xi_0| \widehat{f}(\xi) \big\|_2.$$

Explain how this shows that a function $f \in L^2(\mathbf{R})$ cannot be sharply localised in both physical and Fourier space simultaneously. This is the *uncertainty principle.*

(2*) Extend the result to $\mathbf{R}^n$.

4.

(1) Show that there are numbers $a_n(\xi) \in [0, 1]$ for $n \in \mathbf{Z}_{\geq 1}$, and $\xi \in \mathbf{Z}$ such that $a_n(\xi) \to 1$ for every fixed ξ, and for the sequence of functions

$$g_n(x) = \sum_{\xi \in \mathbf{Z}} a_n(\xi) e^{2\pi i x \xi},$$

$(1_{[-1/2,1/2]} \cdot g_n)_n$ forms an approximate identity.
[Hint: You may want to try out some explicit sequences like $a_n(\xi) = e^{-|\xi|/n}$, or $a_n = 1 - |\xi|/n$ for $|\xi| \leq n$.]

(2*) Extend the result to $\mathbf{T}^d$.

5. Let (X_n) be a sequence of independent random variables taking the values ± 1 with equal probabilities. Let $\lambda \in (0, 1)$ and let $Y = \sum_{n=0}^\infty X_n \lambda^n$. Let ν be the distribution of Y, that is $\nu(A) = \mathbf{P}(Y \in A)$ for all Borel $A \subset \mathbf{R}$.

(1) Prove that

$$\mathbf{E}[e^{-2\pi i Y\xi}] = \prod_{n=0}^{\infty} \cos(2\pi\lambda^n \xi)$$

for all $\xi \in \mathbf{R}$.

(2) Now let $\lambda = \theta^{-1}$, where $\theta = (1+\sqrt{5})/2$ is the golden ratio. Prove that ν is not absolutely continuous with respect to the Lebesgue measure.
[Hint: You may use without proof the fact that $a_k := \theta^k + (-\theta)^{-k}$ is an integer for all $k \in \mathbf{Z}$. If you do not want to take this on blind faith, then show that the sequence satisfies $a_k = a_{k-1} + a_{k-2}$.]

6.

(1) Show that $\mathcal{D}(\mathbf{R}^n)$ is a vector subspace of $\mathcal{S}(\mathbf{R}^n)$. Show that if $\{\varphi_j\}_{j=1}^{\infty}$ is a sequence of compactly supported functions which tends to zero in $\mathcal{D}(\mathbf{R}^n)$ then $\varphi_j \to 0$ in $\mathcal{S}(\mathbf{R}^n)$.

(2) Give an example of a sequence $\{\varphi_j\}_{j=1}^{\infty} \subset C_c^{\infty}(\mathbf{R}^n)$ such that $\varphi_j \to 0$ in $\mathcal{S}(\mathbf{R}^n)$, but φ_j has no limit in $\mathcal{D}(\mathbf{R}^n)$.

7. For each $X \in \{\mathcal{D}(\mathbf{R}^n), \mathcal{S}(\mathbf{R}^n)\}$, suppose $\varphi \in X$ and establish:

(1) If $x_l \in \mathbf{R}^n$, $x_l \to 0$, then

$$\tau_{x_l}\varphi \to \varphi, \qquad \text{in } X \text{ as } l \to \infty,$$

where τ_x is the translation operator defined by $\tau_x\varphi(y) := \varphi(y-x)$.

(2) If $h_l \in \mathbf{R}$, $h_l \to 0$ and $\alpha \in \mathbf{Z}_{\geq 0}^n$ with $|\alpha| = 1$, then

$$\Delta_{h_l}^{\alpha}\varphi \to D^{\alpha}\varphi, \qquad \text{in } X \text{ as } l \to \infty,$$

in X, where $\Delta_h^{\alpha}\varphi := h^{-1}\left[\tau_{-h\alpha}\varphi - \varphi\right]$ is the difference quotient.

8. Let $X \in \{\mathcal{D}(\mathbf{R}^n), \mathcal{S}(\mathbf{R}^n)\}$. For $u \in X'$, $x \in \mathbf{R}^n$, define $\tau_x u$ by $\tau_x u[\varphi] = u[\tau_{-x}\varphi]$ for all $\varphi \in X$. For $\alpha \in \mathbf{Z}_{\geq 0}^n$ with $|\alpha| = 1$, let $\Delta_h^{\alpha} u = h^{-1}\left[\tau_{-h\alpha}u - u\right]$. Show that $\Delta_h^{\alpha} u \to D^{\alpha} u$ as $h \to 0$ in the weak-$*$ topology of X'.

9. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = e^x \cos(e^x)$. Determine whether the corresponding distribution T_f belongs to $\mathcal{D}'(\mathbf{R}), \mathcal{S}'(\mathbf{R})$, respectively.

10.

(1) Suppose $u \in \mathcal{D}'(\mathbf{R})$ satisfies $Du = 0$. Show that u is a constant distribution, i.e. there exists $\lambda \in \mathbf{C}$ such that:

$$u[\varphi] = \lambda \int_{\mathbf{R}} \varphi(x)dx, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbf{R}).$$

[Hint: Fix an appropriate $\varphi_0 \in \mathcal{D}(\mathbf{R})$ and show that any $\varphi \in \mathcal{D}(\mathbf{R})$ may be written as $\varphi(x) = \psi'(x) + c_{\varphi}\varphi_0(x)$ for some $\psi \in \mathcal{D}(\mathbf{R})$, $c_{\varphi} \in \mathbf{C}$.]

(2*) Extend the result to $\mathbf{R}^n$ for $n > 1$.

11. Suppose $u \in \mathcal{D}'(\mathbf{R})$ satisfies $xu = 0$. Show that $u = c\delta_0$ for some $c \in \mathbf{C}$. Find the most general $u \in \mathcal{D}'(\mathbf{R})$ which satisfies $x^k u = 0$ for some $k \in \mathbf{N}$.

12. Suppose $u \in \mathcal{D}'(\mathbf{R}^n)$ is *positive*, i.e. $u[\varphi] \geq 0$ for all $\varphi \in \mathcal{D}(\mathbf{R}^n)$ with $\varphi \geq 0$. Show that u has order 0, i.e., for each $K \subset \mathbf{R}^n$ compact, there is a constant C such that

$$|u[\varphi]| \leq C \sup_{x\in K} |\varphi(x)|, \qquad \text{for all } \varphi \in C_c^{\infty}(K).$$