Lent 2026 **Analysis of Functions – Example Sheet 2** P.P. Varjú

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A supervisor sheet is available on Moodle or by email.

1. Let X be a (real) normed vector space. Show that if $f \in X'$ with $f \neq 0$ then f is an open mapping (i.e. $f(U)$ is open whenever $U \subset X$ is open).

2. Let X be a real normed vector space and suppose $A \subset X$ is open and convex with $0 \in A$. For $x \in X$, define

$$p_A(x) = \inf\{t > 0 : t^{-1}x \in A\}.$$

Show that $p_A(\lambda x) = \lambda p_A(x)$ for all $\lambda \in \mathbf{R}_{>0}$ and $x \in X$, and $p_A(x+y) \leq p_A(x) + p_A(y)$ for all $x, y \in X$. Prove also that $p_A(x) \leq k\|x\|$ for some $k > 0$. Show further that $p_A(x) < 1$ if and only if $x \in A$. [*μ_A is called the Minkowski functional of A.*]

3. Show that there is a bounded linear functional $L : L^\infty(\mathbf{R}) \to \mathbf{C}$ such that $L(f) = f(0)$ for all $f \in C(\mathbf{R}) \cap L^\infty(\mathbf{R})$. Prove that there is no $g \in L^1(\mathbf{R})$ such that

$$L(f) = \int f(x)g(x)dx$$

for all $f \in L^\infty(\mathbf{R})$.

4. Suppose X is a normed vector space, and $V \subset X$ is a closed proper subspace of X and let $0 < \alpha < 1$. Show that there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geq \alpha$ for all $y \in V$. Deduce that the Bolzano–Weiserstrass theorem does not hold if X is an infinite dimensional Banach space.
[The first result above is known as Riesz' Lemma.]

5. For a bounded measurable set $E \subset \mathbf{R}^n$ of positive measure, and any $f \in L^1_{loc.}(\mathbf{R}^n)$, define the mean of f on E to be:

$$⨍_E f(x)dx = \frac{1}{|E|}\int_E f(x)dx.$$

Suppose $1 \leq p < \infty$ and let $(f_j)_{j=1}^\infty$ be a bounded sequence in $L^p(\mathbf{R}^n)$. If $p = 1$, suppose in addition that $\operatorname{supp} f_j \subset B(0,1)$ for all j. Show that $f_j \rightharpoonup f$ for some $f \in L^p(\mathbf{R}^n)$ if and only if

$$⨍_E f_j(x)dx \to ⨍_E f(x)dx$$

for all bounded measurable sets $E \subset \mathbf{R}^n$ of positive measure.

6. Construct a bounded sequence $(f_i)_{i=1}^\infty$ of functions $f_i \in L^1(\mathbf{R})$ such that no subsequence is weakly convergent.

7. Suppose $(H, \langle\cdot,\cdot\rangle)$ is a separable infinite dimensional Hilbert space and let $(u_j)_{j=1}^\infty$ be an orthonormal basis (also called a complete orthonormal system) in H. Let $(x_i)_{i=1}^\infty \subset H$ be a sequence with $\sup\|x_i\| < \infty$.

(a) Show that $x_i \rightharpoonup x$ if and only if $\langle x_i, u_j\rangle \to \langle x, u_j\rangle$ for all $j = 1, 2, \ldots$.
(b) Show there exists a sequence such that $x_i \rightharpoonup 0$, but $x_i \not\to 0$.
(c) Suppose $x_i \rightharpoonup x$. Show that $\|x_i\| \to \|x\|$ if and only if $x_i \to x$.

8. Let X be a separable Banach space, and let $x_1, x_2, \ldots \in B(X)$ be a countable dense subset of $B(X)$. Prove that

$$\mathrm{dist}(f,g) = \sum_{n=1}^{\infty} \frac{|\langle x_n, f-g \rangle|}{2^n}$$

is a metric on $B_{X'}$ and it generates the restriction of the weak-$*$ topology on $B_{X'}$.

9. Let X be a reflexive Banach space, and suppose $Y \subset X$ is a closed subspace. Show that Y is reflexive.

10. Let $T : X \to Y$ be a bounded linear map between two Banach spaces. (You may assume that $X = Y$ and it is a Hilbert space if you wish.)

(a) Let $(x_n) \subset X$ be a sequence such that $x_n \rightharpoonup x$ for some $x \in X$. Prove that $Tx_n \rightharpoonup Tx$.

(b) If T is a compact map, that is $T(B_X)$ is compact in Y for the norm topology, then prove that for every sequence $x_n \rightharpoonup x$, we have $Tx_n \to Tx$.

(c) Now assume that X is a separable Hilbert space and prove that T is a compact map if and only if the property in the previous item holds.

11. Let E be a compact metric space and let $T : E \to E$ be a continuous map. We say that a Borel probability measure μ on E is T-invariant if $\mu(T^{-1}(A)) = \mu(A)$ for every Borel set $A \subset E$.

(a*) Show that a Borel probability measure μ is T-invariant if and only if

$$\int f(Tx)d\mu(x) = \int f(x)d\mu(x)$$

for all $f \in C(E)$.

(b) Let $x_0 \in E$ and let (n_k) be a sequence of integers converging to infinity. For each k, let ν_k be the normalized counting measure of the set $\{x_0, Tx_0, \ldots, T^{n_k-1}x_0\}$. Show that if ν_k converges to a probability measure ν in the weak-$*$ topology, then ν is T invariant.

(c) Conclude that there is always at least one T-invariant Borel probability measure.

(d*) Suppose there is a unique T invariant Borel probability measure μ on E. Show that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \to \int f(x)d\mu(x)$$

for all $f \in C(E)$ and all $x \in E$. Compare this with Birkhoff's pointwise ergodic theorem.

(e) Does the conclusion of item (c) remain true if E is a not necessarily compact metric space?