

ANALYSIS OF FUNCTIONS (PART II)
EXAMPLE SHEET 1

Harder questions highlighted with * and facultative “cultural” questions highlighted with %.
Focus first on questions 1 to 11 in priority for the supervision.

Exercise 1. Let A, B Borel sets of $\mathbb{R}^n$ with finite non-zero measure. Show the convolution $\chi_A * \chi_B(x) := \int_{\mathbb{R}^n} \chi_A(y) \chi_B(x - y) dy$ of their two characteristic functions is a continuous function and is not the zero function. Deduce that $A + B$ contains a ball.

Exercise 2. Let $(A_k)_{k \geq 1}$ Borel sets of $\mathbb{R}^n$ s.t. $\sum_{k \geq 1} \mu(A_k) < +\infty$ (where μ is the Lebesgue measure). Prove that the set of points belonging to infinitely many A_k 's has zero measure.

Exercise 3. Consider a sequence of measurable functions $f_k : \mathbb{R}^n \rightarrow \mathbb{C}$. Prove that the set of points where the sequence converges is measurable.

Exercise 4. Let $f : \mathbb{R}^n \rightarrow [0, +\infty]$ be a measurable function s.t. $\int_{\mathbb{R}^n} f(x) dx = 0$. Prove that f is zero almost everywhere.

Exercise 5. Let $f \in L^1(\mathbb{R}^n)$. Assume that $\int_{\mathbb{R}^n} f(x) \varphi(x) dx = 0$ for all smooth compactly supported functions $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$. Prove that $\|f\|_{L^1(\mathbb{R}^n)} = 0$ and f is zero almost everywhere.

Exercise 6. Let $f \in L^1(\mathbb{R}^n)$. Prove that for any $\varepsilon > 0$ there is $\delta > 0$ s.t. $\int_E |f| dx < \varepsilon$ as soon as $\mu(E) < \delta$ (where μ denotes the Lebesgue measure).

Exercise 7. Give a counter-example showing that the domination assumption is necessary in Lebesgue's dominated convergence Theorem.

***Exercise 8.** Consider $f_n : \mathbb{R}^n \rightarrow \mathbb{R}_+$ a sequence of measurable functions converging pointwise to f on $\mathbb{R}^n$, and s.t. $\int_{\mathbb{R}^n} f_n dx \leq \int_{\mathbb{R}^n} f dx < +\infty$. Prove that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^n} |f_n - f| dx = 0$.

Exercise 9. Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ measurable s.t. $\int_{\mathbb{R}} f dx < +\infty$. Calculate $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} n \ln(1 + (\frac{f}{n})^\alpha) dx$ for $\alpha > 0$.

***Exercise 10.** Construct a sequence of continuous functions $f_n : [0, 1] \rightarrow \mathbb{R}_+$ that converges at no point, and whose integral converges to zero.

Exercise 11. Consider $f : \mathbb{R} \rightarrow [0, +\infty]$ measurable. Recall: Changing variables inside integrals can be used from *Probability & Measure* without proof.

- (1) Assume $\int_{\mathbb{R}} f dx < +\infty$, and define $g(x) := \sum_{k=-\infty}^{+\infty} f(x + k) \in [0, +\infty]$. Prove that $g(x)$ is finite for almost every $x \in \mathbb{R}$.
- (2) Assume f is periodic and has finite integral on any compact set. Prove that $\lim_{n \rightarrow \infty} \frac{1}{n^2} f(nx) = 0$ for almost every $x \in \mathbb{R}$.

Exercise 12. *Approximation of the unit and convolution.*

An *approximation of the unit* is a sequence of measurable functions $\varphi_k : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $k \geq 1$, with $\int_{\mathbb{R}^n} \varphi_k(x) dx = 1$ and support of φ_k included in $B(0, \varepsilon_k)$ for all $k \geq 1$, with $0 < \varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

- (1) Construct an example of such sequence. Can you assume furthermore that the φ_k are smooth?
- (2) Prove that the *translation operator* is continuous in $L^1(\mathbb{R}^n)$: for $f \in L^1(\mathbb{R}^n)$ and $\tau_h f$ defined by $\tau_h f(x) = f(x + h)$, the convergence $\|\tau_h f - f\|_{L^1(\mathbb{R}^n)} \rightarrow 0$ as $h \rightarrow 0$ holds. [Hint: Argue by density of simple functions in $L^1(\mathbb{R}^n)$.]
- (3) Deduce that if $f \in L^1(\mathbb{R}^n)$, the sequence $f * \varphi_k$ is well-defined, belongs to $L^1(\mathbb{R}^n)$ and converges to f in $L^1(\mathbb{R}^n)$.
- (4) Give a new proof of the density of $C_c^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$. [Hint: Truncate f and convolute with a smooth approximation of the unit.]

Exercise 13. Let $I = (0, 1)$, $p \in [1, +\infty)$, $f \in L^p(\mathbb{R})$ where $f = 0$ outside I . Define $f_h(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(y) dy$ for $h > 0$.

- (1) Prove that f_h is well-defined for all $h > 0$.
- (2) Prove that f_h is continuous.
- (3) Prove that $\|f_h\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}$.
- (4) Prove that $\|f_h - f\|_{L^p(\mathbb{R})} \rightarrow 0$ as $h \rightarrow 0$.

Exercise 14. Let E closed set of $\mathbb{R}^n$. The aim is to construct a non-negative smooth function on $\mathbb{R}^n$ s.t. $f(x) = 0$ iff $x \in E$.

- (1) Construct $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ smooth, positive on $B(0, 1)$, that is zero outside $B(0, 1)$ and s.t. $\int_{\mathbb{R}^n} \varphi(x) dx = 1$.
- (2) Let $\varphi_k(x) = \frac{1}{k} \varphi(kx)$, $k \geq 1$. Denote $V_\varepsilon = \{x \mid \text{dist}(x, E) < \varepsilon\}$, where $\text{dist}(x, E) = \inf\{|x - y| \mid y \in E\}$. Let χ_k the characteristic function of $\mathbb{R}^n \setminus V_{2/k}$ and $f_k := \varphi_k * \chi_k$. Prove that f_k is smooth, $f_k \equiv 0$ on $V_{1/k}$, $f_k > 0$ on $\mathbb{R}^n \setminus V_{3/k}$.
- (3) Calculate a supremum bound on all partial derivatives of f_k .
- (4) Conclude by summing the f_k 's.

***Exercise 15.** *Study of the convergence in measure.*

Take a sequence of measurable functions $f_n : [0, 1] \rightarrow \mathbb{C}$. The sequence *converges in measure* to f if for any $\varepsilon > 0$, $\mu(\{x \in \mathbb{R} \mid |f(x) - f_n(x)| > \varepsilon\})$ goes to zero as $n \rightarrow \infty$.

- (1) Prove that if f_n converges almost everywhere to f on $[0, 1]$, it converges in measure to f on $[0, 1]$. Is the reciprocal statement true?
- (2) Prove that if f_n converges to f in $L^p([0, 1])$, $p \in [1, +\infty]$, then it converges in measure to f on $[0, 1]$. Is the reciprocal statement true?
- (3) Prove that if f_n converges in measure to f , then there is a subsequence of f_n converging almost everywhere to f .
- (4) Are the results (1) and (2) true on $\mathbb{R}$ instead of $[0, 1]$?

***Exercise 16.** *Study of the ergodic average.*

Let $p \in (1, +\infty)$ and $f \in L^p((0, +\infty))$, and denote $T[f](x) = \frac{1}{x} \int_0^x f(y) dy$ for $x \in (0, +\infty)$.

- (1) Prove that the function $T[f]$ is well-defined on $(0, +\infty)$.
- (2) Assume f continuous with compact support in $(0, +\infty)$.
 - (a) Give a differential equation expressing f in terms of $T[f]$.
 - (b) Prove $\|T[f]\|_{L^p((0, +\infty))} \leq \frac{p}{p-1} \|f\|_{L^p((0, +\infty))}$. [Hint: Assume first that f is non-negative, and use integration by parts and (b). Relax then the assumption of non-negativity by linearity.]
- (3) Prove that if $f_n \rightarrow f$ in $L^p((0, +\infty))$, then $T[f_n](x)$ converges to $T[f](x)$ for all $x \in (0, +\infty)$.

- (4) Prove $\|T[f]\|_{L^p((0,+\infty))} \leq \frac{p}{p-1} \|f\|_{L^p((0,+\infty))}$ for all $f \in L^p((0,+\infty))$ and compute the norm of the linear map $T : L^p((0,+\infty)) \rightarrow L^p((0,+\infty))$.
- (5) Does T map $L^1((0,+\infty))$ into $L^1((0,+\infty))$?

%Exercise 17. *A theorem due to Lebesgue on Riemann-integrable functions.*

Recall from Riemann's integrability theory: We consider $f : [a, b] \rightarrow \mathbb{R}$ bounded. Let $\mathcal{P} = \{a = t_0 < t_1 < \dots < t_n = b\}$ be a partition of $[a, b]$ and define $\ell_{\mathcal{P}} := \sum_{i=1}^n m_i \chi_{(t_{i-1}, t_i]}$ and $u_{\mathcal{P}} := \sum_{i=1}^n M_i \chi_{(t_{i-1}, t_i]}$, where $m_i := \inf\{f(x) : x \in [t_{i-1}, t_i]\}$ and $M_i := \sup\{f(x) : x \in [t_{i-1}, t_i]\}$. Denote $\int_a^b \ell_{\mathcal{P}} = L(f, \mathcal{P})$ and $\int_a^b u_{\mathcal{P}} = U(f, \mathcal{P})$. Riemann-integrability of f holds when lower and upper Darboux sums both converge to $L(f) = U(f)$ when the mesh size of the partition goes to zero. This limit does not depend on the choice of sequence of partitions, and is called the Riemann integral of f .

- (1) Prove that if $f : [a, b] \rightarrow \mathbb{R}$ bounded is Riemann-integrable, it is measurable and belongs to $L^1([a, b])$.
- (2) Define $H(x) := \inf_{\delta > 0} \sup\{f(y) : y \in [a, b] \text{ and } |y - x| \leq \delta\}$ and $h(x) := \sup_{\delta > 0} \inf\{f(y) : y \in [a, b] \text{ and } |y - x| \leq \delta\}$. Prove that $h(x) \leq f(x) \leq H(x)$ for all $x \in [a, b]$, and f is continuous at x iff $h(x) = H(x)$.
- (3) Consider an increasing (i.e. adding more points at each step) sequence of partitions $\mathcal{P}_k$ with mesh size less than $1/k$, and denote $u = \inf_k u_{\mathcal{P}_k} = \lim_k u_{\mathcal{P}_k}$ and $\ell = \sup_k \ell_{\mathcal{P}_k} = \lim_k \ell_{\mathcal{P}_k}$. Show that these functions are well-defined (including that the limits do exist), satisfy $\ell(x) \leq f(x) \leq u(x)$ for all $x \in [a, b]$, and prove that f Riemann-integrable iff $u = \ell = f$ almost everywhere.
- (4) Denote $N = \cup \mathcal{P}_k$ the set of all points defining the partitions (countable with measure zero). Show that $H(x) = u(x)$ and $h(x) = \ell(x)$ on $x \in [a, b] \setminus N$.
- (5) Deduce that h and H are measurable and prove that f Riemann-integrable iff it is continuous almost everywhere.
- (6) Compare with Lusin's Theorem for measurable functions.

%Exercise 18. *Arzela's dominated convergence Theorem for Riemann-integrable functions.*

- (1) Is the statement of Lebesgue's dominated convergence Theorem correct for bounded functions $f_n : [0, 1] \rightarrow \mathbb{R}$ when " L^1 " is replaced by "Riemann-integrable"?
- (2) Consider a nonincreasing sequence of continuous functions $p_n : [0, 1] \rightarrow \mathbb{R}$ ($p_{n+1} \leq p_n$) that converges pointwise to zero, prove that the convergence is uniform.
- (3) Consider $f_n : [0, 1] \rightarrow \mathbb{R}$ Riemann-integrable with $|f_n| \leq 1$ and converging pointwise to zero, and define $g_n(x) := \sup_{m \geq n} |f_m(x)|$. Given $\varepsilon > 0$, construct h_n continuous s.t. $0 \leq h_n \leq g_n$ and $L(g_n) \leq \int_0^1 h_n + \frac{\varepsilon}{2^n}$ (the lower Darboux integral $L(g_n)$ is defined in the previous exercise).
- (4) Define $p_n(x) = \min\{h_1(x), \dots, h_n(x)\}$ and prove using (2) that $\int_0^1 p_n$ goes to zero as $n \rightarrow \infty$.
- (5) Prove that $h_n \leq p_n + \sum_{j=1}^{n-1} (g_j - h_j)$.
- (6) Deduce that $\limsup_{n \rightarrow \infty} \int_0^1 h_n \leq \varepsilon$.
- (7) Deduce that $\limsup_{n \rightarrow \infty} L(g_n) \leq \varepsilon$ and finally $\limsup_{n \rightarrow \infty} \int_0^1 |f| \leq \varepsilon$.
- (8) Prove the following theorem: *Consider a sequence of bounded Riemann-integrable functions $f_n : [0, 1] \rightarrow \mathbb{R}$ that converges pointwise to a bounded Riemann-integrable function f and satisfies $|f_n| \leq F$ with F bounded Riemann-integrable, then the Riemann integral $\int_0^1 f_n \rightarrow \int_0^1 f$ as $n \rightarrow \infty$.*