Lent Term 2024 O. Randal-Williams

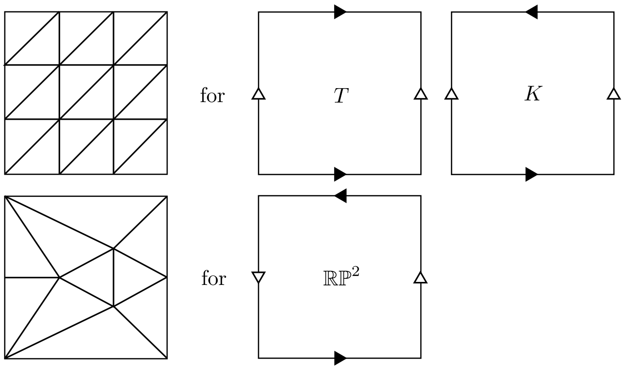
II Algebraic Topology // Example Sheet 3

1. An *abstract simplicial complex* consists of a finite set V_X (called the *vertices*) and a collection X (called the *simplices*) of subsets of V_X such that if $\sigma \in X$ and $\tau \subseteq \sigma$, then $\tau \in X$. A map $f : (V_X, X) \to (V_Y, Y)$ of abstract simplicial complexes is a function $f : V_X \to V_Y$ such that $f(\sigma) \in Y$ for all $\sigma \in X$.

   (i) For a simplicial complex K in $\mathbb{R}^m$, show that the *abstraction* of K,
   $$V_X = \{0\text{-simplices of } K\} \qquad X = \{\{a_0, \ldots, a_n\} \subset V_X \mid \langle a_0, \ldots, a_n\rangle \in K\}$$
   is an abstract simplicial complex. Show that any abstract simplicial complex arises in this way (up to isomorphism), and that if simplicial complexes K and L have isomorphic abstractions, then $|K|$ and $|L|$ are homeomorphic.

   (ii) Show that there exists an infinite sequence of points $(x_1, x_2, \ldots) \in \mathbb{R}^m$ such that any $(m+1)$ of them are affinely independent. Hence show that if (V_X, X) is an abstract simplicial complex with all simplices of dimension $\leq n$, then there is a simplicial complex K in $\mathbb{R}^{2n+1}$ with abstraction isomorphic to (V_X, X).

2. Show that there are triangulations of the torus, Klein bottle, and projective plane as follows:



   How many vertices, edges and faces does each triangulation have? What is the number $\chi =$ vertices $-$ edges $+$ faces for each triangulation?

3. Use the simplicial approximation theorem to show that

   (i) if K and L are simplicial complexes, there are at most countably many homotopy classes of continuous maps $f : |K| \to |L|$,

   (ii) if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map,

   (iii) for any vertex v of a simplicial complex K the based map $(|K_{(2)}|, v) \to (|K|, v)$ (i.e. the inclusion of the 2-skeleton) induces an isomorphism on fundamental groups. Show how to prove the same result using the Seifert–van Kampen theorem.

4. Let K be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex. Show that $L = K \setminus \{\sigma\}$ is again a simplicial complex, and that the inclusion $V_L \to V_K$ defines a simplicial map $i : L \to K$.

   If σ has dimension n, note that $d_n(\sigma)$ is a $(n-1)$-cycle and consists of simplices of L, so represents a class $[d_n(\sigma)] \in H_{n-1}(L)$; this defines a homomorphism $\varphi : \mathbb{Z} \to H_{n-1}(L)$ by $1 \mapsto [d_n(\sigma)]$. Construct a homomorphism $\phi : H_n(K) \to \mathbb{Z}$ such that
   $$0 \longrightarrow H_n(L) \xrightarrow{i_*} H_n(K) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\varphi} H_{n-1}(L) \xrightarrow{i_*} H_{n-1}(K) \longrightarrow 0$$

is exact (i.e. the image of one map is *precisely* the kernel of the next), and show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for $j \neq n-1, n$.

5. Let K be a simplicial complex, and suppose that $\sigma \in K$ is not a proper face of any simplex, and that $\tau \leq \sigma$ is a face of one dimension lower which is not a face of any other simplex. Show that $L = K \setminus \{\sigma, \tau\}$ is again a simplicial complex, and that the inclusion $V_L \to V_K$ defines a simplicial map $i : L \to K$.

   (i) By constructing a chain homotopy inverse to $i_\bullet : C_\bullet(L) \to C_\bullet(K)$, show that $i_* : H_j(L) \to H_j(K)$ is an isomorphism for all j.

   (ii) Prove the same thing using the previous question (twice) instead.

6. Using the two previous questions, compute the homology groups of the simplicial complexes described in Q2, and describe generators for each homology group.

7. Let K be an n-dimensional simplicial complex such that

   (i) every $(n-1)$-simplex is a face of precisely two n-simplices, and

   (ii) every pair of n-simplices can be connected by a sequence of n-simplices such that adjacent terms share an $(n-1)$-dimensional face.

   Show that $H_n(K)$ is either $\mathbb{Z}$ or trivial. In the first case show $H_n(K)$ is generated by a cycle which is a sum of all n-simplices with suitable orientations.

8. Suppose that $(C_\bullet, d)$ is a chain complex of finitely-generated free abelian groups with $H_*(C_\bullet, d) = 0$. Show that $id : C_\bullet \to C_\bullet$ is chain homotopic to the zero map.

Optional Questions

9. For simplicial complexes K and L inside $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, show that $|K| \times |L| \subset \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ is the polyhedron of a simplicial complex. *[Hint: Prove it first in the case where both K and L consist of a single simplex (plus all its faces).]*

Comments or corrections to `or257@cam.ac.uk`