

Algebraic Topology, Examples 2

Michaelmas 2020

The *wedge* of two spaces $X \vee Y$ is the quotient space obtained from the disjoint union $X \sqcup Y$ by identifying two points $x \in X$ and $y \in Y$.

Questions marked with a (*) are optional.

1. A *graph* X is defined as follows. Consider two discrete spaces V and E with continuous maps $\iota, \tau : E \rightarrow V$. Then

$$X = (V \sqcup (E \times I)) / \sim$$

where \sim is the smallest equivalence relation so that $(e, 0) \sim \iota(e)$ and $(e, 1) \sim \tau(e)$.

- (a) Show that S^1 is homeomorphic to a graph.
 - (b) Show that $S^1 \vee S^1$ is homeomorphic to a graph.
 - (c) Draw all the covering spaces of S^1 of degree 2 or 3.
 - (d) Draw all the covering spaces of $S^1 \vee S^1$ of degree 2 or 3.
2. Prove that every covering space of a graph is a graph.
 3. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = 1$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite).

- (a) Show that the quotient map $X \rightarrow G \backslash X$ is a covering map.
 - (b) Deduce that if X is simply-connected then for any point $[x] \in G \backslash X$ we have an isomorphism of groups $\pi_1(G \backslash X, [x]) \cong G$.
 - (c) Hence show that for any $m \geq 2$ there is a space X with fundamental group \mathbb{Z}/m and universal cover S^3 . [*Hint: Consider S^3 as the unit sphere in \mathbb{C}^2 . You may use without proof the fact that S^3 is simply connected.*]
4. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the usual two loops. Describe covering spaces associated to:
 - (a) $\langle\langle a \rangle\rangle$, the normal subgroup generated by a ;
 - (b) $\langle a \rangle$, the subgroup generated by a ;

- (c) the kernel of the homomorphism $\phi : \langle a, b \rangle \rightarrow \mathbb{Z}/4\mathbb{Z}$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

5. Show that the groups

$$G = \langle a, b \mid a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle$$

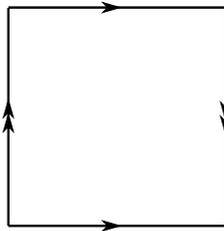
are isomorphic. Show that this group is non-abelian and infinite.

[Hint: Construct surjective homomorphisms to appropriate groups.]

6. What is the universal cover of $S^1 \vee S^2$?
7. Let $X = S^1 \vee S^1$ as in question 4. Consider the space Y obtained from X by attaching 2-cells along loops in the homotopy classes a^2 and b^2 , so that

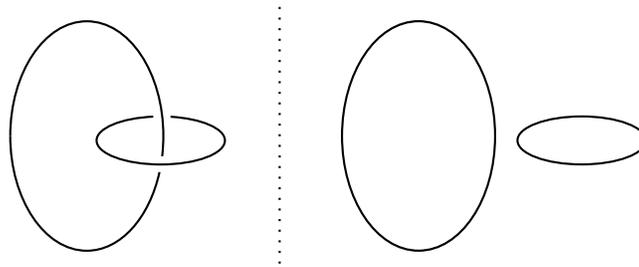
$$\pi_1(Y, x_0) \cong \langle a, b \mid a^2, b^2 \rangle.$$

- (a) Construct (in pictures) the covering space of Y corresponding to the subgroup $\langle a \mid a^2 \rangle$.
- (b) Construct (in pictures) the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \mid a^2, b^2 \rangle \rightarrow \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\text{Ker}(\phi)$ is isomorphic to $\langle a, b \mid a^2, b^2 \rangle$.
8. The *Klein bottle* is the surface obtained from the following identification diagram.



Explain how to construct the Klein bottle by attaching a 2-cell to a graph. Deduce that its fundamental group has a presentation $\langle a, b \mid baba^{-1} \rangle$, and show this is isomorphic to the group in Q13 of Sheet 1.

9. Consider the following configurations of pairs of circles in S^3 (we have drawn them in \mathbb{R}^3 ; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

10. (*) In question (1d) you drew various covering maps $p: \widehat{Y} \rightarrow S^1 \vee S^1$. Which of these arose from the construction in Question 3? That is, for which \widehat{Y} is there a group G acting freely and properly discontinuously so that p is the quotient map $\widehat{Y} \rightarrow G \backslash \widehat{Y}$?
11. (*) A *tree* is a simply connected graph. A *star* is a tree with a vertex x_0 such that one end of each edge is attached to x_0 . A *leaf* of a tree is a vertex attached to only one edge. Prove that every tree is homotopy equivalent to a star, relative to its leaves.
12. (*) A tree T which is a subgraph of a graph X is *maximal* if it contains every vertex. You may assume that every graph has a maximal tree.
- If $T \subset G$ is a tree, show that the quotient map $G \rightarrow G/T$ is a homotopy equivalence. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex. [*Hint: Use question 11.*]
 - Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. [*We have not required that a graph have finitely many edges.*]
 - Deduce that every subgroup of a free group is free.