Algebraic Topology, Examples 2

Michaelmas 2017

1. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = e$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \varnothing\}$ is finite).

(a) Show that the quotient map $X \to X/G$ is a covering map.

(b) Deduce that if X is simply-connected and locally path-connected then for any point $[x] \in X/G$ we have an isomorphism of groups $\pi_1(X/G, [x]) \cong G$.

(c) Hence show that for any $m \geq 2$ there is a space X with fundamental group $\mathbb{Z}/m$ and universal cover S^3. *[Hint: Consider S^3 as the unit sphere in $\mathbb{C}^2$.]*

[You may use the fact that S^3 is simply connected without proof.]

2. Show that the inclusion $i : (S^1 \times \{1\}) \cup (\{1\} \times S^1) \hookrightarrow S^1 \times S^1$ does not admit a retraction. [As usual, think of $S^1 \subset \mathbb{C}$, the elements of unit modulus, containing 1.]

3. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b\rangle$ where a and b are given by the two characteristic loops. Describe covering spaces associated to:

(a) $\langle\langle a \rangle\rangle$, the normal subgroup generated by a;

(b) $\langle a \rangle$, the subgroup generated by a;

(c) the kernel of the homomorphism $\phi : \langle a, b\rangle \to \mathbb{Z}/4$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

Show that the free group on two letters contains a copy of itself as a proper subgroup.

4. Consider the 2-dimensional cell complex Y obtained from X in the previous question by attaching 2-cells along loops in the homotopy classes a^2 and b^2, so that

$$\pi_1(Y, x_0) \cong \langle a, b \,|\, a^2, b^2\rangle.$$

(a) Construct (in pictures) the covering space of Y corresponding to the subgroup $\langle a \,|\, a^2\rangle$.

(b) Construct (in pictures) the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \,|\, a^2, b^2 \rangle \to \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\mathrm{Ker}(\phi)$ is isomorphic to $\langle a, b \,|\, a^2, b^2 \rangle$.

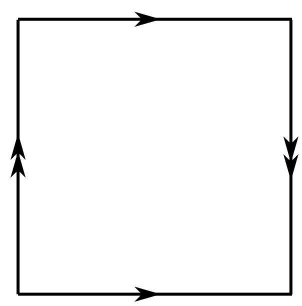
5. Show that the groups

$$G = \langle a, b \,|\, a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \,|\, xyxy^{-1}x^{-1}y^{-1} \rangle$$

are isomorphic. Show that this group is non-abelian and infinite.

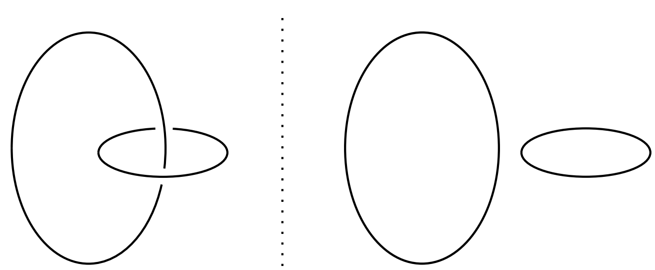
[Hint: Construct surjective homomorphisms to S_3 and $\mathbb{Z}$.]

6. The *Klein bottle* is the surface obtained from the following identification diagram.



Show that the Klein bottle has a cell structure with a single 0-cell, two 1-cells, and a single 2-cell. Deduce that its fundamental group has a presentation $\langle a, b \,|\, baba^{-1} \rangle$, and show this is isomorphic to the group in Q13 of Sheet 1.

7. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity).



By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.

8. Let $f : X \to Y$ be a continuous map. The *mapping cylinder* of f is the space

$$M_f := ((X \times [0,1]) \sqcup Y) \,/\, \sim$$

where $\sim$ is the finest equivalence relation such that $(x,1) \sim f(x)$. Let ξ be the unique map from X to a point. Assuming that Y is contractible, show that there is a pair of homotopy equivalences

$$M_f \xrightarrow{\phi} M_\xi \ ; \ M_\xi \xrightarrow{\psi} M_f$$

so that $\phi \circ \psi$ is homotopic to the identity *relative to* $X \times \{0\} \subseteq M_\xi$, and $\psi \circ \phi$ is homotopic to the identity *relative to* $X \times \{0\} \subseteq M_f$.

9. A *graph* G is a 1-dimensional cell complex. A *tree* is a graph which is contractible. A tree T inside a graph G is *maximal* if no strictly larger subgraph is a tree. You may assume that every graph has a maximal tree.

(a) If $T \subset G$ is a tree, show that the quotient map $G \to G/T$ is a homotopy equivalence. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex. *[Hint: Use question 8.]*

(b) Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. *[We have* not *required that a graph have finitely many edges.]*

(c) Deduce that a subgroup of a free group is free. *[You may use without proof the fact that a covering space of a graph is again a graph.]*