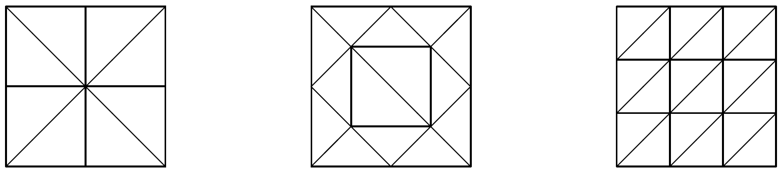
# Algebraic Topology Examples 3

**PTJ Mich. 2011**

Starred questions are not necessarily harder than the unstarred ones (which are, in any case, not all equally difficult), but they go beyond what you need to know for the course. Comments and corrections are welcome, and should be sent to `ptj@dpmms.cam.ac.uk`.

**1**. (i) If we regard the projective plane $\mathbb{R}P^2$ as the quotient of the unit square obtained by identifying opposite points around its boundary, which (if any) of the following pictures represents a triangulation of $\mathbb{R}P^2$?



*(ii) What is the smallest possible number of 2-simplices in a triangulation of $\mathbb{R}P^2$?

**2**. Show that it is possible to choose an infinite sequence of points $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)$ in $\mathbb{R}^m$ which are *in general position* in the sense that no $m+1$ of them lie in a proper affine subspace (i.e., a coset of a proper vector subspace). Deduce that if $K$ is an abstract simplicial complex having no simplices of dimension greater than $n$, it is possible to find a *geometric realization* of $K$ (that is, a space having a triangulation isomorphic to $K$) which is a subspace of $\mathbb{R}^{2n+1}$. [This result is best possible; it can be shown that the $n$-skeleton of a $(2n+2)$-simplex cannot be realized in $\mathbb{R}^{2n}$.]

**3**. Use the Simplicial Approximation Theorem to show

(i) that if $X$ and $Y$ are polyhedra then there are only countably many homotopy classes of continuous maps $X \to Y$;

(ii) that if $m < n$ then any continuous map $S^m \to S^n$ is homotopic to a constant map.

**4**. Show that the fundamental group of a polyhedron depends only on its 2-skeleton: that is, for any simplicial complex $K$ and vertex $a$ of $K$, we have $\Pi_1(|K|, a) \cong \Pi_1(|K_{(2)}|, a)$ where $K_{(2)}$ is the 2-skeleton of $K$. [Apply the Simplicial Approximation Theorem to paths in $|K|$ and homotopies between them. Question 10 below provides an explicit combinatorial method of calculating the fundamental group of a polyhedron, based on this idea.]

**5**. Let $K$ and $L$ be simplicial complexes. Construct a triangulation of $|K| \times |L|$. [Method: take the points $(\hat\sigma, \hat\tau)$, where $\sigma$ is a simplex of $K$ and $\tau$ is a simplex of $L$, as vertices, and say that a sequence

$$((\widehat{\sigma_0}, \widehat{\tau_0}), (\widehat{\sigma_1}, \widehat{\tau_1}), \ldots, (\widehat{\sigma_n}, \widehat{\tau_n}))$$

spans a simplex if and only if we have $\sigma_{i-1} \leq \sigma_i$ and $\tau_{i-1} \leq \tau_i$ for each $i$, at least one of the two inequalities being proper. You may find it helpful to consider first what this gives when $|K|$ and $|L|$ are both 1-simplices.]

**6**. Two simplicial maps $h, k \colon K \rightrightarrows L$ are said to be *contiguous* if, for each simplex $\sigma$ of $K$, there is a simplex $\sigma*$ of $L$ such that both $h(\sigma)$ and $k(\sigma)$ are faces of $\sigma*$. Show that

(i) any two simplicial approximations to a given map $f \colon |K| \to |L|$ are contiguous;

(ii) any two contiguous maps $K \rightrightarrows L$ induce homotopic maps $|K| \rightrightarrows |L|$;

(iii) if $f, g \colon |K| \rightrightarrows |L|$ are any two homotopic maps, then for a suitable subdivision $K^{(r)}$ of $K$ there exists a sequence of simplicial maps $h_1, \ldots, h_n \colon K^{(r)} \to L$ such that $h_1$ is a simplicial approximation to $f$, $h_n$ is a simplicial approximation to $g$ and each pair $(h_i, h_{i+1})$ is contiguous. [Method: let $H$ be a homotopy between $f$ and $g$, and show that for sufficiently large $n$ and $r$ the mappings $x \mapsto H(x, \frac{i-1}{n})$ and $x \mapsto H(x, \frac{i}{n})$ have a common simplicial approximation defined on $K^{(r)}$, for each $i \leq n$.]

**7**. Let $K$ be a (geometric) simplicial complex in $\mathbb{R}^m$. The *suspension* $SK$ of $K$ is the complex in $\mathbb{R}^{m+1}$ whose vertices are those of $K$ (regarded as lying in $\mathbb{R}^m \times \{0\}$) and the two points $\mathbf{v}_\pm = (0, \ldots, 0, \pm 1)$, and whose simplices are those of $K$ together with those spanned by the vertices of a simplex of $K$ together with one or the other (but not both) of the new vertices.

(i) Verify that $SK$ is a simplicial complex, and show in particular that if $|K| \cong S^n$ then $|SK| \cong S^{n+1}$.

(ii) Let $h_r \colon C_r(K) \to C_{r+1}(SK)$ be the homomorphism sending an $r$-simplex $\sigma$ of $K$ to the chain $\langle \mathbf{v}_+, \sigma\rangle - \langle \mathbf{v}_-, \sigma\rangle$. Verify that $dh_r = h_{r-1}d$ if $r \geq 1$, and deduce that $h_r$ induces a well-defined homomorphism $h_* \colon H_r(K) \to H_{r+1}(SK)$. [Later we'll see that $h_*$ is an isomorphism.]

**8**. Let $X$ be the 'complete $n$-simplex', i.e. the simplicial complex formed by an $n$-simplex together with all its faces. What is the rank of the chain group $C_k(X)$? Show that, for $1 \leq k < n$, the $k$th homology group of the $k$-skeleton $X_{(k)}$ of $X$ is free abelian of rank $\binom{n}{k+1}$. [Hint: calculate the ranks of the groups $Z_k(X)$ for all $k$, using the fact that $H_k(X) = 0$ for all $k > 0$.]

**9**. Let $K$ be a simplicial complex satisfying the following conditions:

(i) $K$ has no simplices of dimension greater than $n$;

(ii) Every $(n-1)$-simplex of $K$ is a face of exactly two $n$-simplices;

(iii) For any two $n$-simplices $\sigma$ and $\tau$ of $K$, there exists a finite sequence of $n$-simplices, beginning with $\sigma$ and ending with $\tau$, in which each adjacent pair of simplices have a common $(n-1)$-dimensional face.

Show that $H_n(K)$ is either $\mathbb{Z}$ or the trivial group, and that in the former case it is generated by a cycle which is the sum of all the $n$-simplices of $K$, with suitable orientations.

*__10__. Let $K$ be a simplicial complex. We define an *edge path* in $K$ to be a finite sequence $(a_0, a_1, \ldots, a_n)$ of vertices of $K$ such that $(a_i, a_{i+1})$ spans a simplex for each $i$. An *edge loop* is an edge path such that $a_0 = a_n$; the *product* of two edge paths $(a_0, \ldots, a_n)$ and $(a_n, \ldots, a_m)$ is $(a_0, \ldots, a_n, \ldots, a_m)$. Two edge paths are said to be *equivalent* if one can be converted into the other by a finite sequence of moves of the form: replace $(\ldots, a_i, a_{i+1}, a_{i+2}, \ldots)$ by $(\ldots, a_i, a_{i+2}, \ldots)$ provided $\{a_i, a_{i+1}, a_{i+2}\}$ spans a simplex of $K$ (or the inverse of this move). (We allow the possibility that $a_i, a_{i+1}, a_{i+2}$ may not all be distinct; thus, for example, we may always replace $(\ldots, a_i, a_{i+1}, a_i, \ldots)$ by $(\ldots, a_i, a_i, \ldots)$, and we may further replace this by $(\ldots, a_i, \ldots)$ provided there is at least one other vertex in the sequence.) Show that equivalence classes of edge loops based at $a_0$ form a group $E(K, a_0)$, and use the Simplicial Approximation Theorem (plus question 6) to show that $E(K, a_0) \cong \Pi_1(|K|, a_0)$.

*__11__. Let $K$ be a simplicial complex, and $a$ a vertex of $K$. Show that there is a homomorphism $h \colon \Pi_1(|K|, a) \to H_1(K)$. [$h$ is called the *Hurewicz homomorphism*; to construct it, observe that an edge path in $K$, as defined in the previous question, can be thought of as an 'ordered sum' of oriented 1-simplices, whereas a 1-chain is an unordered sum of such simplices.] Show also that $h$ is surjective if $K$ is connected; and that if $f \colon K \to L$ is a simplicial map sending $a$ to $b$ then the diagram

$$\begin{array}{ccc} \Pi_1(|K|, a) & \xrightarrow{\;h\;} & H_1(K) \\ \Big\downarrow f_* & & \Big\downarrow f_* \\ \Pi_1(|L|, b) & \xrightarrow{\;h\;} & H_1(L) \end{array}$$

commutes. [It can be shown that the kernel of $h$ is exactly the commutator subgroup of $\Pi_1(|K|)$ — that is, for connected $K$, $H_1(K)$ is isomorphic to the largest abelian quotient group of $\Pi_1(|K|)$.]

*__12__. Consider the quotient space $X$ of $S^3$ constructed in question 12(iv) on sheet 2. Use the previous question to show that $H_1(X) = 0$. [It can be shown that $X$ has the same homology groups as $S^3$, but it is not homotopy equivalent to $S^3$.]