Local Path-Connectedness — An Apology

PTJ Lent 2011

For around 40 years I have believed that the two possible definitions of local path-connectedness, as set out in question 14 on the first Algebraic Topology example sheet, are not equivalent. This belief has been reinforced by the many topology textbooks which insist that the first, less 'natural', definition is the right one to use; not a few of them further reinforce the impression by calling it 'semi-local path-connectedness', which is clearly meant to imply that it is strictly weaker than the 'natural' definition.

Some 25 years ago, I discovered the equivalence of the formally weaker definition with the openness of the canonical map $\mathrm{Cts}(I, X) \to X \times X$ sending u to $(u(0), u(1))$; for reasons which I don't need to go into here, that seemed to me a good and sufficient explanation of why it was the right definition to use (presuming the two to be inequivalent). At the time I wondered at the fact that no-one had bothered to explain this reason to me when I was a student; and I resolved that I would be more honest with my students if I ever found myself lecturing on algebraic topology in future.

Therefore, when preparing example sheets at the start of this term, I initially put it on example sheet 1 as (the first half of) question 14. I then thought that I could make the question a bit easier by replacing $u \mapsto (u(0), u(1))$ by the mapping $u \mapsto u(0)$, so I changed it. Unfortunately, whilst this does make the implication in one direction easier to prove, it makes the converse false, as can be seen by considering a space X which is totally disconnected but not discrete (for example, $X = \mathbb{Q}$).

It also occurred to me that I didn't actually know a counterexample to separate the two definitions (and none of the textbooks that I then consulted provided one). I though about it for a while, and succeeded in convincing myself that a modification of the counterexample that I knew for question 4(ii) would do the trick; so I added the second half of question 14.

More recently, I happened to look up something in the textbook *Topology* (Ellis Horwood, 1988) by my old friend Ronnie Brown, who was for many years Professor of Pure Maths at Bangor in north Wales, but is now retired. (Ronnie is a lovable eccentric: I wouldn't recommend his textbook to anyone other than a high first-class student, since it's likely to be far more confusing than helpful to anyone else.) I happened to notice his definition of local path-connectedness: he dutifully gives the 'unnatural' definition first, but then immediately proves that it is equivalent to the 'natural' one. Moreover, the proof of equivalence is carried over unchanged from the first edition of Ronnie's book, published in 1968 by McGraw–Hill. **So why didn't anyone tell me that 40 years ago?**

Here, with my sincere apologies to anyone who may have wasted time looking for a counterexample, is Ronnie's proof. Suppose X satisfies the 'weaker' definition of local path-connectedness. Let $x \in X$, and let U be an open neighbourhood of x. Let U' be the path-component of x in U (i.e., the set of points which can be joined to x by paths in U). If $y \in U'$, then U is an open neighbourhood of y, so by the 'weaker' definition there exists an open V with $y \in V \subseteq U$ such that y can be joined to any point of V by a path in U. Since we can paste these paths on to a path from x to y, we clearly have $V \subseteq U'$; so U' is a neighbourhood of each of its points, i.e. it is open. And U' is path-connected by definition; so the path-connected open sets form a base for the topology.