ALGEBRAIC TOPOLOGY **LENT 2010**

EXAMPLE SHEET 1

1. Let $a : S^n \to S^n$ be the antipodal map ($a(x) = -x$.) If n is odd, show that a is homotopic to the identity map. (Hint: try $n = 1$ first.)

2. Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that $f(x) = -x$ for some $x \in S^1$.

3. Which of the letters $A, B, \ldots, Z$ are contractible? Which are homotopy equivalent to S^1?

4. Let X be a contractible space, and let Y be any space. Show that
   (a) X is path connected.
   (b) $X \times Y$ is homotopy equivalent to Y.
   (c) any two maps from Y to X are homotopic.
   (d) if Y is path connected, any two maps from X to Y are homotopic.

5. Show that the torus minus a point, the Klein bottle minus a point, and $\mathbb{R}^2$ minus two points are all homotopy equivalent to $S^1 \vee S^1$. (Hint: draw pictures showing how $S^1 \vee S^1$ can be embedded as a deformation retract in each space. Describe the retraction in words or pictures, rather than with formulas.)

6. Embed S^k in S^n ($k < n$) as the set $\{(x_1, x_2, \ldots, x_{k+1}, 0, 0, \ldots, 0) \mid \sum x_i^2 = 1\}$. Show that the complement $S^n - S^k$ is homotopy equivalent to S^{n-k-1}.

7. Show that the cylinder $S^1 \times I$ and the Mobius band both have fundamental group isomorphic to $\mathbb{Z}$.

8. Regarding S^1 as the unit complex numbers, describe the induced homomorphisms $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ when
   (a) $f(e^{i\theta}) = e^{i(\theta+\pi/2)}$.
   (b) $f(e^{i\theta}) = e^{in\theta}$ for some $n \in \mathbb{Z}$.
   (c) $f(e^{i\theta}) = \begin{cases} e^{i\theta} & \text{if } 0 \le \theta \le \pi \\ e^{i(2\pi-\theta)} & \text{if } \pi \le \theta \le 2\pi \end{cases}$

9. Suppose (X, x) and (Y, y) are spaces with basepoints. Show that

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

10. Suppose G is a topological group; *i.e.* G is a topological space with a group structure such that the multiplication map $m : G \times G$, $m(a, b) = ab$ and the inverse map $\iota : G \to G$, $\iota(a) = a^{-1}$ are both continuous. For any pair of loops γ_1, γ_2 based at the identity e, let $\gamma_1 \cdot \gamma_2$ be the loop given by $\gamma_1 \cdot \gamma_2(s) = m(\gamma_1(s), \gamma_2(s))$. Show that $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$ (products in π_1) are both homotopic to $\gamma_1 \cdot \gamma_2$. Conclude that $\pi_1(G, e)$ is abelian.

11.* Suppose G is a topological group which is path-connected and locally path connected, *i.e.* if U is open and G and $p \in U$, there is a path connected open set $U' \subset U$ with $p \in U'$. Given a connected covering map $p : G' \to G$ and an element e' in $p^{-1}(e)$, show that there is a unique group structure on G' for which e' is the identity and p is a homomorphism.

12.* Some examples of topological groups:

(a) Let $G = SO(2)$, the group of 2×2 orthogonal matrices with determinant 1. Show that $G \simeq S^1$.

(b) Let $G = SU(2)$, the group of 2×2 unitary matrices of determinant 1. Show that $G \simeq S^3$. (In fact, S^1 and S^3 are the only spheres which admit the structure of a topological group.)

(c) Let $G = SO(3)$, the group of 3×3 orthogonal matrices with determinant 1. Show that $G \simeq \mathbb{RP}^3$. (Hint: the set of 180° rotations is homeomorphic to $\mathbb{RP}^2$.)

(d) The set $\{\pm I\}$ is a normal subgroup of $SU(2)$. As suggested by part (c), show that the quotient $SU(2)/\pm I$ is isomorphic to $SO(3)$.

13.* Suppose that (X, d) is a metric space, and give

$$\mathrm{Map}(S^1, X) = \{f : S^1 \to X \mid f \text{ is continuous}\}$$

the *uniform metric*, for which $\overline{d}(f, g) = \max_{x \in S^1} d(f(x), g(x))$. Check that if d, d' are two metrics on X which induce the same topology on X, then $\overline{d}$ and $\overline{d}'$ induce the same topology on $\mathrm{Map}(S^1, X)$.

Given $H : S^1 \times [0, 1] \to X$, define $f_t(x) = H(x, t)$. Show that H is continuous if and only if the map $[0, 1] \to Map(S^1, X)$ given by $t \mapsto f_t$ is continuous.

J.Rasmussen@dpmms.cam.ac.uk