ALGEBRAIC TOPOLOGY **MICHELMAS 2008**

EXAMPLE SHEET 3

1. Suppose X and Y are simplicial complexes with vertices x and y respectively. Let $X \vee Y$ be the one point union obtained by identifying x and y. For $i > 0$, show that $H_i(X \vee Y) \cong H_i(X) \oplus H_i(Y)$. What about H_0?

2. Let X be a simplicial complex consisting of the four sides of a square and its two diagonals. (So X has 5 vertices and 8 edges.) Compute $H_*(X)$. How is this related to problem 1?

3. Let X be the space obtained by removing two points from T^2. What is $H_*(X)$?

4. Show that $\mathbb{RP}^2$ can be given the structure of a weakly simplicial complex with 2 vertices, 3 edges, and 2 2-simplices. Use this to compute $H_*(\mathbb{RP}^2)$. (Hint: $\mathbb{RP}^2 = S^1 \cup_f B^2$, where $f : S^1 \to S^1$ is given by $f(z) = z^2$.)

5. Use the simplicial approximation theorem to show that any $f : S^n \to S^m$ $(n < m)$ is null-homotopic.

6. If X is a topological space, define the *cone* on X to be the space $CX = X \times [0, 1]/\sim$, where $(x, 1) \sim (y, 1)$ for all $x, y \in X$. (*i.e.* we collapse $X \times 1$ to a single point.)
   (a) Show that CX is contractible.
   (b) If X is a simplicial complex, show that CX can be given the structure of a simplicial complex with one new vertex v_0 and a new $(n+1)$-dimensional simplex for each n-dimensional simplex of X.
   (c) Let $C_*(CX)$ be the chain complex of this simplicial complex. Show that the inclusion $i : C_*(v_0) \to C_*(CX)$ is a chain homotopy equivalence.

7. Let $X = \Delta^n$ be the n-simplex, and let $C_*(\Delta^n)$ be the associated chain complex.
   (a) What is the rank of the group $C_k(\Delta^n)$?
   (b) Using problem 5, show that $H_k(\Delta^n) = 0$ for $k > 0$, and that $H_0(\Delta^n) \cong \mathbb{Z}$.
   (c) Use the homeomorphism $S^{n-1} \simeq \partial\Delta^n$ to compute

$$H_i(S^{n-1}) \cong \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

   (d) More generally, let X_k be the k-*skeleton* of Δ^n, that is the union of all the i-dimensional faces of Δ^n for $i \leq k$. Compute $H_*(X_k)$.

8. Show that any finite simplicial complex X is homeomorphic to a subcomplex of Δ^{v-1}, where v is the number of vertices of X.

9. Let X be a simplical complex, and let $X_2 \subset X$ be its 2-skeleton, *i.e.* the union of all the 0, 1, and 2–dimensional simplices of X.

   (a) Use the simplicial approximation theorem to show $\pi_1(X_2, x) \cong \pi_1(X, x)$.

   (b) Use the Seifert van-Kampen theorem to prove the same result.

10.* Let A be a 2×2 matrix with integer coefficients. Multiplication by A defines a linear map $L_A : \mathbb{R}^2 \to \mathbb{R}^2$.

   (a) Show that L_A descends to a well-defined map $f_A : T^2 \to T^2$.

   (b) Compute the induced map $f_{A*} : H_*(T^2) \to H_*(T^2)$.

   (c) Show that f_A is a homeomorphism if and only if the induced map on H_2 is an isomorphism.

11.* Suppose (C, d) is a chain complex defined over a field F; *i.e.* $C = \bigoplus C_i$, where each C_i is a vector space over F, and $d_i : C_i \to C_{i-1}$ is an F–linear map with $d_i \circ d_{i+1} = 0$. Let $(H_i(C), 0)$ be the chain complex whose groups are the homology groups of C, and with trivial differential. Show that (C, d) is chain homotopy equivalent to $(H_i(C), 0)$. If we replace F by $\mathbb{Z}$, show that the corresponding statement is false.

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