ALGEBRAIC TOPOLOGY (PART II)
EXAMPLE SHEET 3

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(1) Let X and Y be triangulable spaces. Show that $H_i(X \vee Y) \cong H_i(X) \oplus H_i(Y)$ for $i > 0$. Show that for a wedge of n circles,

$$H_i(S^1 \vee \cdots \vee S^1) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^n & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) Compute the homology groups of the two-holed torus, that is, the torus with two points removed.

(3) Compute the homology groups and Betti numbers of the space obtained from the torus $S^1 \times S^1$ by identifying $S^1 \times \{p\}$ to a point and $S^1 \times \{q\}$ to a point, for two distinct points p and q in S^1.

(4) Let K and K' be simplicial complexes and $A, B \subset K$, $A', B' \subset K'$ subcomplexes. Let $f : K \to K'$ be a simplicial map that sends A into A' and B into B'.

(a) Use the definition of the Mayer-Vietoris homomorphism Δ_* to show that the following diagram commutes.

$$\begin{array}{ccc} H_{n+1}(K) & \xrightarrow{\Delta_*} & H_n(A \cap B) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H_{n+1}(K') & \xrightarrow{\Delta_*} & H_n(A' \cap B') \end{array}$$

(b) Show that if the homomorphisms $H_n(A \cap B) \to H_n(A' \cap B')$, $H_n(A) \to H_n(A')$, and $H_n(B) \to H_n(B')$ are isomorphisms for all n, then the homomorphisms $H_n(K) \to H_n(K')$ are isomorphisms for all n.

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(5) The antipodal map $f\colon S^n \to S^n$, $f(x) = -x$.

(a) Let K be a simplicial complex such that $|K|$ is homeomorphic to the sphere S^{n-1}. By adding two vertices make a new complex L such that $|L|$ is homeomorphic to the sphere S^n.

(b) For $n \geq 1$, show that the antipodal map $S^n \to S^n$ induces on $H_n(S^n)$ the map multiplication by $(-1)^{n+1}$.

(c) Show that if n is even, the antipodal map on S^n is not homotopic to the identity.

(6) (a) Let $f : X \to Y$ be a finite covering map where finiteness means that $f^{-1}\{y\}$ is finite for $y \in Y$. Suppose that Y has a triangulation; this always induces a triangulation of X. Show that $\chi(X) = \deg(f)\chi(Y)$. Do the Betti numbers always satisfy $b_i(X) = \deg(f)b_i(Y)$?

(b) Let G be a finite group acting on S^n freely, that is, each $x \in S^n$ has a neighborhood U such that $U \cap g(U) = \emptyset$ for any $1 \neq g \in G$. Show that G has order at most 2 if n is even. (Assume that the quotient space is triangulable.)

(7) Let K be a simplicial complex and $f : K \to K$ be a simplicial self-map. Choose an order for the vertices of K (which determines an orientation of each simplex in K). Let C_i denote the vector space of i-chains with coefficients in $\mathbb{R}$ (for each $i \geq 0$) and $C_i f : C_i \to C_i$ the induced linear transformation.

(a) Use the standard basis for C_i (of i-simplices of K) to show that if the trace $tr(C_i f)$ is not zero, then $f\colon |K| \to |K|$ has a fixed point.

(b) Show that $tr(C_i f) = tr(Z_i f) + tr(B_{i-1} f)$ and $tr(Z_i f) = tr(B_i f) + tr(H_i f)$. Here Z_i denotes the vector space of i-cycles (with coefficients in $\mathbb{R}$) and $Z_i f$ is the induced map on the cycles; $B_{i-1} \cong C_i/Z_i$ and $B_{i-1} f$ is the induced map on B_{i-1}; and $H_i = H_i(K, \mathbb{R}) = Z_i/B_i$ and $H_i f$ is the induced map on H_i.

(c) Show that

$$\sum (-1)^i tr(C_i f) = \sum (-1)^i tr(H_i f).$$

(d) Conclude that if $\sum (-1)^i tr(H_i f)$ is not zero, then $f\colon |K| \to |K|$ has a fixed point.

This is a weak version of the Lefschetz fixed point theorem. It only takes uniform continuity, plus the simplicial approximation theorem, plus some bookkeeping to prove the full version:

Theorem (The Lefschetz fixed point theorem). Let X be a compact triangulable space and let $f : X \to X$ be a continuous map. If the Lefschetz number $\Lambda(f) = \sum(-1)^i tr(H_i f)$ is not zero, then f has a fixed point.

(See for example M. A. Armstrong, *Basic Topology*, section 9.4 for an argument.)

What makes this theorem particularly powerful is that the formula for $\Lambda(f)$ only depends on the homotopy class of the map. (Also note: the Euler characteristic is the Lefschetz number of the identity map.)

(8) An n-dimensional pseudomanifold is a simplicial complex K with the following properties.

(i) Every simplex is a subsimplex of an n-simplex.

(ii) Every $(n-1)$-simplex is a face of exactly two n-simplices.

(iii) For any two n-simplices σ and τ, there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_r = \tau$ such that each σ_i and σ_{i+1} intersect along an $(n-1)$-simplex.

We remark that any triangulation of a connected n-manifold is an n-dimensional pseudomanifold.

Let K be an n-dimensional pseudomanifold. Show that $H_n(K)$ is isomorphic to either 0 or $\mathbb{Z}$. (One can say that K is orientable in the latter case.) If $H_n(K) = \mathbb{Z}$, find an n-chain in $Z_n(K)$ generating $H_n(K)$.

(9) (a) Compute the Euler characteristic of all closed surfaces.

(b) Show that the Klein bottle is homeomorphic to the connected sum of two copies of the real projective plane.