ALGEBRAIC TOPOLOGY (PART II)
EXAMPLE SHEET 1

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(1) Show that a connected manifold is path-connected. (For the purpose of this problem, we can define an n-dimensional manifold to be a topological space such that every point has an open neighbourhood homeomorphic to an open subset of Euclidean space $\mathbb{R}^n$.)

(2) Let $f : S^1 \to S^1$ be a map which is not homotopic to the identity map. Show that $f(x) = -x$ for some $x \in S^1$.

(3) Let $f \colon X \to S^n$ be a map which is not surjective. Prove that f is null-homotopic, that is, it is homotopic to a constant map.

(4) Let CX be the cone on a space X. By definition, this is the quotient space (also called identification space) of $X \times [0, 1]$ obtained by identifying all the points $(x, 0)$ for $x \in X$ to the same point. Show that CX is contractible.

(5) Classify the capital letters $A, B, \ldots, Z$ up to homeomorphism and also up to homotopy type. Of course, it depends on how you write them.

(6) Show that there is a deformation retract of $B^2 \times I$ onto $S^1 \times I \cup B^2 \times \{0\}$.

(7) For a map $f : S^1 \to X$, let $X \cup_f B^2$ be the quotient space obtained from the disjoint union of X and B^2 by identifying each point x on the boundary S^1 of B^2 to the point $f(x)$ in X. Show that if $f, g : S^1 \to X$ are homotopic maps, then the spaces $X \cup_f B^2$ and $X \cup_g B^2$ are homotopy equivalent.

(8) Show that the Möbius strip and the cylinder $S^1 \times [0, 1]$ both have fundamental group isomorphic to $\mathbb{Z}$. (We can define the

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Möbius strip as the space made from the square $[0,1] \times [0,1]$ by identifying the point $(0,t)$ with $(1,1-t)$ for all t, although it is probably more useful to draw a picture.)

(9) Let $X = \mathbb{R}^n - \{P\}$ where P is a point. What is $\pi_1(X)$?

(10) Show that the fundamental group of the product of two path-connected spaces is the product of their fundamental groups.

(11) Describe the homomorphism of fundamental groups induced by the quotient map $S^1 \to \mathbb{RP}^1$.

(12) Show that the fundamental group of real projective space $\mathbb{RP}^n$, $n \geq 2$, is generated by the image of a great-circle path in S^n from the north to the south pole.

(13) Prove Brouwer fixed point theorem in dimension two, that is, prove that for any map $f\colon B^2 \to B^2$ there is $x \in B^2$ such that $f(x) = x$.