Examples sheet 3 for Part II Algebraic Topology

Burt Totaro

13 November 2006

(1) Let X and Y be triangulable spaces, and choose basepoints $x \in X$, $y \in Y$. Show that $\widetilde{H}_*(X \vee Y) \cong \widetilde{H}_*(X) \oplus \widetilde{H}_*(Y)$. Show that for a wedge of n circles,

$$H_i(S^1 \vee \cdots \vee S^1) = \begin{cases} \mathbf{Z} & \text{if } i = 0 \\ \mathbf{Z}^n & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) For X the torus, the sphere, the two-holed torus, and the projective plane, X can be defined by making the following identifications of edges on the boundary of a 2-disc: $ABA^{-1}B^{-1}$, AA^{-1}, $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$, AA (respectively). Assume that X has a triangulation $X = Y \cup Z$ where Y is homeomorphic to a smaller closed disc in the interior of D^2 and $Y \cap Z$ is homeomorphic to S^1. Use the Mayer-Vietoris sequence to compute the homology of X for these four surfaces X.

(3) Let K and K' be simplicial complexes and $A, B \subset K$, $A', B' \subset K'$ subcomplexes. Let $f : K \to K'$ be a simplicial map that sends A into A' and B into B'.

(a) Use the definition of the Mayer-Vietoris boundary homomorphism to show that the following diagram commutes.

$$\begin{array}{ccc} H_{n+1}(K) & \xrightarrow{\partial} & H_n(A \cap B) \\ {\scriptstyle H_{n+1}(f)}\downarrow & & \downarrow{\scriptstyle H_n(f)} \\ H_{n+1}(K') & \xrightarrow{\partial} & H_n(A' \cap B') \end{array}$$

(b) Show that if the maps $H_i(A \cap B) \to H_i(A' \cap B')$, $H_i A \to H_i A'$, and $H_i B \to H_i B'$ are isomorphisms for all i, then the maps $H_i K \to H_i K'$ are isomorphisms for all i.

(4) The antipodal map on S^n.

(a) Let K be a simplicial complex homeomorphic to the sphere S^{n-1}. Show that the suspension of K is homeomorphic to the sphere S^n.

(b) Show that the antipodal map $S^n \to S^n$ induces on $H_n(S^n)$ the map multiplication by $(-1)^{n+1}$.

(c) Show that if n is even, the antipodal map on S^n is not homotopic to the identity.

(5) (a) Let $f : X \to Y$ be a finite covering space of topological spaces. Suppose that Y has a triangulation; this always induces a triangulation of X. Show that $\chi(X) = \deg(f)\chi(Y)$. Do the Betti numbers always satisfy $b_i(X) = \deg(f) b_i(Y)$?

(b) Show that any finite group acting freely on the n-sphere, for n even, must have order at most 2. (Assume that the quotient space is triangulable.)

(6) Let K be a simplicial complex and $f : K \to K$ be a simplicial self-map. Choose an order for the vertices of K (which determines an orientation of each simplex in K). Let C_i denote the vector space of i-chains with coefficients in $\mathbf{R}$ (for each $i \geq 0$) and $C_i f : C_i \to C_i$ the induced linear transformation.

(a) Use the standard basis for C_i (of i-simplices of K) to show that if the trace $\mathrm{tr}(C_i f)$ is not zero, then the geometric realization of f has a fixed point.

(b) Show that $\mathrm{tr}(C_i f) = \mathrm{tr}(Z_i f) + tr(B_{i-1} f)$ and $tr(Z_i f) = tr(B_i f) + tr(H_i f)$. Here Z_i denotes the vector space of i-cycles (with coefficients in $\mathbf{R}$) and $Z_i f$ is the induced map on the cycles; $B_{i-1} \cong C_i / Z_i$ and $B_{i-1} f$ is the induced map on B_{i-1}; and $H_i = H_i(K, \mathbf{R}) = Z_i / B_i$ and $H_i f$ is the induced map on H_i.

(c) Show that

$$\sum (-1)^i tr(C_i f) = \sum (-1)^i tr(H_i f).$$

(d) Conclude that if $\sum (-1)^i tr(H_i f)$ is not zero, then the geometric realization of f has a fixed point.

This is a weak version of the Lefschetz fixed point theorem. It only takes uniform continuity, plus the simplicial approximation theorem, plus some bookkeeping to prove the full version:

Theorem (The Lefschetz fixed point theorem). Let X be a compact triangulable space and let $f : X \to X$ be a continuous map. If the Lefschetz number $\Lambda(f) = \sum (-1)^i \mathrm{tr}(H_i f)$ is not zero, then f has a fixed point.

(See for example M. A. Armstrong, *Basic Topology*, section 9.4 for an argument.)

What makes this theorem particularly powerful is that the formula for $\Lambda(f)$ only depends on the homotopy class of the map. (Also note: the Euler characteristic is the Lefschetz number of the identity map.)

(7) Using the computation of the simplicial complex $\partial \Delta^{n+1}$, prove:

(a) Euclidean spaces of different dimensions are not homeomorphic.

(b) The disc D^{n+1} does not retract onto the sphere S^n. (This is part of the proof of the Brouwer fixed point theorem for D^{n+1}.)

(8) An n-dimensional pseudomanifold is a simplicial complex K with the following properties.

(i) Every simplex is a subsimplex of an n-simplex.

(ii) Every $(n-1)$-simplex is a face of exactly two n-simplices.

(iii) For any two n-simplices σ and τ, there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_r = \tau$ such that each σ_i and σ_{i+1} intersect along an $(n-1)$-simplex.

We remark that any triangulation of a connected n-manifold is an n-dimensional pseudomanifold.

(a) Give an example of a 2-dimensional pseudomanifold whose geometric realization is not a 2-manifold.

(b) Let K be an n-dimensional pseudomanifold. Show that $H_n(K)$ is isomorphic to either 0 or $\mathbf{Z}$. (One can say that K is orientable in the latter case.) If $H_n(K)$ is isomorphic to $\mathbf{Z}$, show that it is generated by the sum of the n-simplices with signs.

(9) (a) List all the closed surfaces, orientable or not, and compute their Euler characteristics.

(b) Show that the Klein bottle is homeomorphic to the connected sum of two copies of the real projective plane.