Examples sheet 2 for Part II Algebraic Topology

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(1) Compute the fundamental group of the wedge $S^1 \vee S^2$. Draw the universal cover of this space.

(2) (a) Show that the fundamental group of the complement of a finite set in the plane is a free group.

(b) Show that the complement of a finite set in 3-space is simply connected.

(c) Compute the fundamental group of the complement of the circle $\{(x_0, x_1, x_2, x_3) \in S^3 : x_0 = x_1 = 0\}$ in the 3-sphere S^3. (One approach: use that S^3 minus the point $(0, 0, 0, 1)$ is homeomorphic to $\mathbf{R}^3$, by stereographic projection.)

(3) Show that the group $\langle a, b | a^3 = 1, b^2 = 1, bab^{-1} = a^2 \rangle$ is isomorphic to the symmetric group S_3.

(4) Let (X, x) be a based space, $f : S^1 \to X$ a based map, and $Y = X \cup_f D^2$ the space obtained by gluing D^2 to X along f.

(a) Let $\alpha \in \pi_1(X, x)$ be the element represented by f. Show that $\pi_1(Y, x)$ is isomorphic to $\pi_1(X, x)/N$, where N is the normal subgroup of $\pi_1(X, x)$ generated by α.

(b) Use (a) to show that every finitely presented group is the fundamental group of some space.

(5) Let L be the "infinite ladder of circles" given by the subset of $\mathbf{R}^2$ consisting of the union of the circle of radius 1/2 around each point $(n, 0)$ for $n \in \mathbf{Z}$.

(a) Choose a basepoint for L and show that the fundamental group is a free group on a (countably) infinite number of generators.

(b) Show that the action of $\mathbf{Z}$ on L (where $n \in \mathbf{Z}$ acts by $(x, y) \mapsto (x+n, y)$ is free and its quotient is homeomorphic to $S^1 \vee S^1$. (Freeness of a group action is as defined in lectures; some people call this property "free and properly discontinuous".)

(c) Conclude that the free group on an infinite number of generators is isomorphic to a subgroup of the free group on two generators.

(6) Use the simplicial approximation theorem to show:

(a) If X and Y are compact triangulable spaces, then there are at most countably many homotopy classes of maps from X to Y.

(b) If $m < n$, then every map $S^m \to S^n$ is homotopic to a constant map.

(7) Let X be a simplicial complex.

(a) Show that if $|X|$ is connected, then any two vertices in X can be connected by a sequence of edges in X.

(b) Let X_2 be the 2-skeleton of X, the subcomplex containing all vertices, 1-simplices, and 2-simplices. For a vertex a, show that $\pi_1(X_2, a) \to \pi_1(X, a)$ is an isomorphism.

(8) Let $n \geq 1$. Assume that the homology of the $(n+1)$-simplex $\Delta[n+1]$ satisfies

$$H_i(\Delta[n+1]) = \begin{cases} \mathbf{Z} & i = 0 \\ 0 & i > 0. \end{cases}$$

Use the homeomorphism $S^n \cong \partial\Delta[n+1]$ to prove that

$$H_i(S^n) = \begin{cases} \mathbf{Z} & i = 0, n \\ 0 & i \neq 0, n. \end{cases}$$

(9) For each of the following exact sequences of abelian groups, say what you can about the unknown group A and/or the unknown homomorphism α.

(a) $0 \to \mathbf{Z}/2 \to A \to \mathbf{Z} \to 0$

(b) $0 \to \mathbf{Z} \to A \to \mathbf{Z}/2 \to 0$

(c) $0 \to \mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}/2 \to 0$

(d) $0 \to A \to \mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \to \mathbf{Z}/2 \to 0$

(e) $0 \to \mathbf{Z}/3 \to A \to \mathbf{Z}/2 \to \mathbf{Z} \xrightarrow{\alpha} \mathbf{Z} \to 0$

(10) The Five Lemma

Consider the following commutative diagram of abelian groups, where the rows are exact.

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \xrightarrow{f} & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Suppose that the maps $A \to A'$, $B \to B'$, $D \to D'$, and $E \to E'$ are isomorphisms. Show that the middle map $\gamma : C \to C'$ must be an isomorphism, as follows.

(a) First show that $C \to C'$ is injective: Take an element x in C which maps to 0 in C'. (1) Show that x maps to 0 in D and hence that x is the image of some $y \in B$. (2) Show that y is in the image of A and conclude that $x = 0$.

(b) Now that that $C \to C'$ is onto. Take an element $x' \in C'$ and show that it is in the image of C as follows. (1) Show that there is an element $z \in C$ such that $\gamma(z)$ and x' in C' have the same image in D'. (2) Show that there is an element $y \in B$ whose image in B' maps to $x' - \gamma(z)$. Conclude that $\gamma(z + f(y)) = x'$.