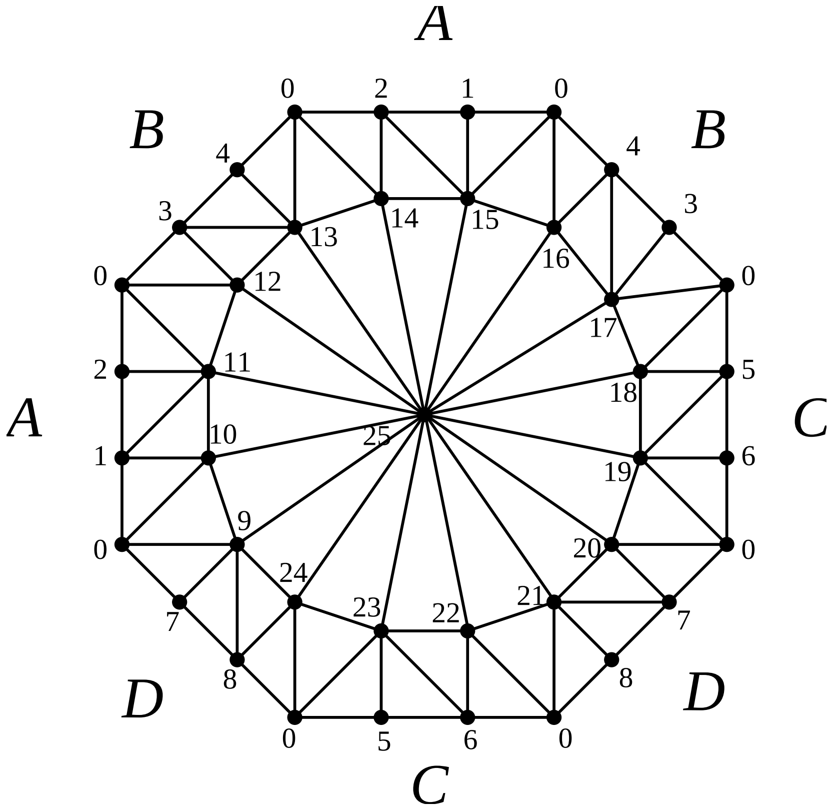
Algebraic Topology 2004 **Example Sheet 3x**

Sample Solution

Here is a sample solution for problem 7 for the two-holed torus, which is by far the most complicated of the examples (but the ordinary torus is still too complicated to compute blindly with matrices). The computation depends on the triangulation chosen for problem 1. Here is one possibility:



There is nothing in particular that is special about this triangulation – it's just the first one I happened to draw. It might be that there is another one that would make this problem easier. If you picked a triangulation that doesn't posses any sort of pattern, I would guess that it could make the problem a lot harder.

Since we only have 0, 1, and 2 simplices, we have that the homology is zero except in degrees 0, 1, and 2.

Computing $H_2 \cong \mathbb{Z}$:

Since $C_3 = 0$, we just need to compute Z_2, and then $H_2 = Z_2$. Let

$$z = \sum a_{v_0,v_1,v_2}\, c_{v_0,v_1,v_2}$$

where the sum ranges over the set of 2-simplices $\{v_0, v_1, v_2\}$, and suppose $dz = 0$.

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The signs in this problem are easier if instead of following our vertex order for the simplices in the sum above, we go clockwise around each 2-simplex in the picture, e.g., using $c_{1,11,10}$ rather than $c_{1,10,11}$ $(= -c_{1,11,10})$.

First we'll show that each a_{v_0,v_1,v_2} is determined by $a_{0,1,10}$. Then we'll show that there is a cycle with $a_{0,1,10} = 1$ (which is then necessarily unique). It will follow that every cycle is a multiple of the one with $a_{0,1,10} = 1$, and $H_2 \cong \mathbb{Z}$.

Since we have assumed that $dz = 0$, we must have $a_{1,11,10} = a_{0,1,10}$ since $\{0, 1, 10\}$ and $\{1, 10, 11\}$ are the only 2-simplices that have the face $\{1, 10\}$ – the differential on $c_{1,11,10}$ gives a summand $-c_{1,10}$ and the differential of $c_{0,1,10}$ gives a summand of $c_{1,10}$. Likewise, we must have $a_{1,2,11} = a_{1,10,11}$, $\ldots$, $a_{0,10,9} = a_{7,0,9}$. Looking at the remaining 2-simplices, we must have $a_{9,10,25} = a_{0,10,9} = a_{0,1,10}$, and continuing around clockwise, we have $a_{10,11,25} = a_{9,10,25}, \ldots$, $a_{23,24,25} = a_{22,23,25}$. Thus, if z is a cycle, then the coefficients of all the 2-simplices are determined by $a_{0,1,10}$.

Finally, consider the chain x with $a_{0,1,10} = 1$ and the rest of the coefficients as above $(1 = a_{1,11,10} = \cdots = a_{0,10,9} = a_{9,10,25} = \cdots = a_{23,24,25})$. Then by the analysis above, in dx the coefficients of the "inner edges" (the 1-simplices not making up the edges A, B, C, D) are all zero. For the "outer edges", starting at the vertex 0 on the copy of A on the left, each outer edge $\{v, w\}$ occurs in exactly two 2-simplices: one is v, w, p for some $p \in \{9, \ldots, 24\}$ and the other is w, v, q for some $q \in \{9, \ldots, 24\}$. It follows that in dx, the coefficient of $c_{v,w}$ is also zero, and so $dx = 0$.

Computing $H_0 \cong \mathbb{Z}$:

(We'll return to H_1 below.)

Every 0-chain is a zero cycle, so $H_0 = C_0/B_0$. Since $dc_{0,1} = c_1 - c_0$, we have that c_0 and c_1 represent the same element of H_0. Likewise looking at each of the edges around the outside, we see that $c_2, \ldots, c_8$ all represent the same element as c_0 in H_0. Since $dc_{0,9} = c_9 - c_0$, c_9 represents the same element as c_0 in H_0, and looking at the edges in the $9, 10, \ldots, 24$ circle, we see that $c_{10}, \ldots, c_{24}$ all represent the same element as c_0 in H_0. Finally, since $dc_{9,25} = c_{25} - c_9$, we see that c_{25} represents the same element as c_0 in H_0. It follows that H_0 is generated by the image of c_0.

To see that $H_0 \cong \mathbb{Z}$, consider the homomorphism $f\colon C_0 \to \mathbb{Z}$ that takes a 0-chain $\sum a_i\, c_i$ to $\sum a_i$. Since for any 1-simplex $\{v, w\}$, $f(dc_{v,w}) = f(c_w - c_v) = 0$, f is zero on the boundaries. It follows that f factors through H_0. Since $f(c_0) = 1$ and H_0 is generated by c_0, H_0 must be the free on c_0. (And the homomorphism f is an isomorphism.)

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Computing $H_1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$:

The more refined answer is that H_1 is the free abelian group on the cycles "A", "B", "C" and "D":

$$\begin{aligned}
A &= c_{0,1} + c_{1,2} + c_{2,0}, \\
B &= c_{0,3} + c_{3,4} + c_{4,0}, \\
C &= c_{0,5} + c_{5,6} + c_{6,0}, \\
D &= c_{0,7} + c_{7,8} + c_{8,0}.
\end{aligned}$$

First we need to show that these generate H_1.

Let z be a 1-cycle. First we show that z represents the same element in H_1 as a 1-cycle z' where the coefficients of $c_{9,25}, \ldots, c_{24,25}$ are all zero. Suppose z has coefficients $a_{9,25}, \ldots, a_{24,25}$ for these, and let

$$\begin{aligned}
z' = z &+ a_{9,25}\, dc_{9,10,25} + (a_{10,25} - a_{9,25})\, dc_{10,11,25} \\
&+ (a_{11,25} - a_{10,25} + a_{9,25})\, dc_{11,12,25} \\
&+ \cdots + (a_{24,25} - a_{23,25} + \cdots - a_{9,25})\, dc_{24,9,25}.
\end{aligned}$$

It's clear that z and z' represent the same element of H_1. By construction, the coefficients of $c_{10,25}, \ldots, c_{24,25}$ in z' are zero, and the coefficient of $c_{9,25}$ is $a_{24,25} - a_{23,25} + \cdots + a_{10,25}$. On the other hand, since the differential of z' is zero and these are the only generators whose differential contains a summand of c_{25}, we must have that the coefficient of $c_{9,25}$ is zero.

Next observe that z' represents the same element in homology as a 1-cycle z'' with the coefficients of $c_{9,10}, \ldots, c_{24,9}$ also all zero – we just add multiples of $c_{0,10,9}$, $c_{1,11,10}$, $c_{0,12,11}$, … to cancel them.

Next, for z'' as above, the coefficient of $c_{0,10}$ must be equal to minus the coefficient of $c_{1,10}$ since the coefficient of c_{10} in dz'' is zero, so add that multiple of $dc_{0,1,10}$ to get a new cycle (representing the same element of homology) with the coefficient of $c_{0,10}$ also zero. Next add a multiple of $dc_{1,2,11}$ to make the coefficient of $c_{1,11}$ also zero, and in the resulting cycle, we must have that the coefficient of $c_{2,11}$ is minus the coefficient of $c_{0,11}$ (since the coefficient of c_{11} in the cycle's differential is zero); we add a multiple of $dc_{2,0,11}$. By the same argument (seven more times), we can go around the outer ring adding boundaries, until we get a cycle z''' (representing the same element of H_1) where non-zero coefficients only occur for (possibly) $c_{0,1}$, $c_{1,2}$, $c_{2,0}$, $c_{0,3}$, $c_{3,4}$, $c_{4,0}$, $c_{0,5}$, $c_{5,6}$, $c_{6,0}$, $c_{0,7}$, $c_{7,8}$, and $c_{8,0}$. Looking at the coefficient of c_1 in dz''', we see that the coefficients of $c_{0,1}$ and $c_{1,2}$ must be equal. Looking at the coefficient of c_2 in dz''', we see that the coefficients of $c_{1,2}$ and $c_{2,0}$ must be equal. Likewise, the coefficients of $c_{0,3}$, $c_{3,4}$, $c_{4,0}$ must be equal to each other (but not nec. to those of $c_{0,1}$, $c_{1,2}$, $c_{2,0}$), the coefficients of $c_{0,5}$, $c_{5,6}$, $c_{6,0}$ must be equal, and the coefficients of $c_{0,7}$, $c_{7,8}$, $c_{8,0}$ must be equal. Thus, z''' is a linear combination of A, B, C, and D.

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Next we need to see that they are independent: If $pA + qB + rC + sD = 0$ (in H_1) for integers p, q, r, s, then $p = q = r = s = 0$. So suppose there is a 2-chain x with $dx = pA + qB + rC + sD$ (in C_1).

Proof 1.

Write $x = \sum a_{v_0,v_1,v_2} c_{v_0,v_1,v_2}$. Let $a = a_{1,2,11}$. Looking at the coefficient of $c_{2,11}$ in dx, we must have $a_{2,0,11} = a$. Looking at the coefficient of $c_{0,11}$ in dx, we must have that $a_{0,12,11} = a$. looking at the coefficient of $c_{0,12}$ in dx, we must have that $a_{0,3,12} = a$. And so on. We see that the coefficient for any 2-simplex in the outer ring is a. We have $p = a_{1,2,11} - a_{2,1,15}$, $q = a_{3,4,13} - a_{4,3,17}$, $r = a_{5,6,19} - a_{6,5,23}$, and $s = a_{7,8,21} - a_{8,7,9}$, which must then all be zero.

Proof 2.

A, B, C, D are generators for homology with coefficients in $\mathbb{R}$ (by the same argument as above) and the equation $dx = pA + qB = rC + sD$ still holds in $C_1(K, \mathbb{R})$. So it suffices to show that $H_1(K, \mathbb{R})$ is four dimensional. (The rest of this paragraph is in terms of real coefficients.) We have that the kernel of $d: C_2 \to C_1$ is 1-dimensional and C_2 is 56 dimensional (by counting the 2-simplices), so the image of the differential, B_1, is 55 dimensional. We have that C_1 is 84-dimensional (by counting the 1-simplices). We have that C_0 is 26-dimensional, and so the image of the differential $d: C_1 \to C_0$ must be 25-dimensional (since the quotient is 1-dimensional), and so the kernel of the differential, Z_1, must be $84 - 25 = 59$ dimensional. It follows that Z_1/B_1 is $59 - 55 = 4$ dimensional.

Proof 3.

The preceding two proofs worked well in this case, but don't work as well when there is torsion (when the homology is not a free abelian group). Here is a trick that works more generally and can be adapted to the case of $\mathbb{P}^2$, where you would look for an appropriate homomorphism with target $\mathbb{Z}/2$.

Consider the homomorphism $\alpha: C_1 \to \mathbb{Z}$ that takes a 1-chain $\sum a_{v_0,v_1} c_{v_0,v_1}$ to the integer

$$a_{1,2} + a_{1,11} + a_{10,11} - a_{11,25} - a_{12,25} - a_{13,25} - a_{14,25} - a_{14,15} - a_{2,15}$$

Looking at 2-simplices c_{v_0,v_1,v_2}, it is easy to check that $\alpha(dc_{v_0,v_1,v_2}) = 0$ – the only ones where it could possibly be non-zero are $c_{1,2,11}$, $c_{1,11,10}$, $c_{10,11,25}$, $c_{11,12,25}$, $c_{13,14,25}$, $c_{14,15,25}$, $c_{2,15,14}$, and $c_{2,1,15}$, which can be checked by hand. So $\alpha(dx) = 0$. But $\alpha(A) = 1$, $\alpha(B) = 0$, $\alpha(C) = 0$, and $\alpha(D) = 0$, so $p = 0$.

The same kind of trick defines homomorphisms β, γ, and δ which are zero on boundaries and are one on exactly one of B, C, D and are zero on the remaining ones and also on A. Applying these functions to the equation $dx = pA + qB + rC + sD$ then shows that $q = 0$, $r = 0$, and $s = 0$.

End of Example Sheet 3x.

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