Algebraic Topology 2004 **Example Sheet 3s**

Supplement

$1\frac{1}{2}$. Here is a more careful articulation of the definition of triangulation. Recall that the standard n-simplex $\Delta[n]$ has vertices $\{0,\ldots,n\}$; a *subsimplex* of $\Delta[n]$ is a (affine) map $\Delta[i] \to \Delta[n]$ induced by an order preserving injection $\{0,\ldots,i\} \to \{0,\ldots,n\}$ – so $\Delta[n]$ has precisely 2^{n+1} subsimplices (including the non-proper subsimplex $\mathrm{id}\colon \Delta[n] \to \Delta[n]$). If $\sigma\colon \Delta[n] \to X$ is a map, we will call the composite $\Delta[i] \to \Delta[n] \to X$ a subsimplex of σ. We write $|\sigma|$ for the image of σ in X and $|\mathring{\sigma}|$ for the image of $\Delta[n] - \partial\Delta[n]$ (note $\partial\Delta[0]$ is empty). With these clarifications, the definition from class (repeated below) is now precise.

Definition Let X be a compact hausdorff space. A (finite) triangulation on X consists of a finite set $\mathcal{T}$ of maps $\sigma\colon \Delta[n] \to X$ that are homeomorphisms onto their images, such that:

(i) If $\sigma\colon \Delta[n] \to X$ is in $\mathcal{T}$, then every subsimplex of σ is in $\mathcal{T}$.

(ii) Every element of X is in $|\mathring{\sigma}|$ for a unique $\sigma \in \mathcal{T}$.

(iii) If $|\sigma| \cap |\tau|$ is non-empty for $\sigma, \tau \in \mathcal{T}$, then there exists $\rho \in \mathcal{T}$ that is a subsimplex of σ and a subsimplex of τ and that satisfies $|\rho| = |\sigma| \cap |\tau|$.

Show that the underlying combinatorial structure of $\mathcal{T}$ is a simplicial complex:

(a) Let $V = \{\sigma\colon \Delta[0] \to X \mid \sigma \in \mathcal{T}\}$, and let $S \subset \mathcal{P}V$ be

$$S = \{\{a_0,\ldots,a_n\} \subset V \mid \text{There exists } \sigma \in \mathcal{T} \text{ such that } a_0,\ldots,a_n \text{ are subsimplexes of } \sigma\}.$$

Show that V, S defines a simplicial complex

(b) Show that the elements of S are in one-to-one correspondence with the elements of Σ (say $A \mapsto \sigma_A$) such that $A \subset B$ if and only if σ_A is a subsimplex of σ_B.

$3\frac{1}{3}$. Infinite triangulations. This problem explores the idea of infinite simplicial complexes and infinite triangulations and for this one problem (only) we drop our convention that simplicial complex means finite simplicial complex and triangulation means finite triangulation. The definition of a possibly infinite simplicial complex is just like the definition of a (finite) simplicial complex except that we do not assume the set of vertices is finite:

Definition A possibly infinite simplicial complex consists of a set V (called the vertices) and a set S of finite subsets of V (called the simplices) such that

(i) S contains all the singleton sets

(ii) If $\sigma \in S$, then every subset of σ is in S

For a simplicial complex $K = (V, S)$, we construct the standard model as follows. Let $\mathbb{R}^V$ be the real vector space of formal finite linear combinations of elements of V (with coefficients in $\mathbb{R}$) with inner product defined by having

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the elements of V orthogonal of unit length. Topologize $\mathbb{R}^V$ by declaring that a $U \subset \mathbb{R}^V$ is open if and only if $U \cap \mathbb{R}^{V_0}$ is open for every finite subset $V_0 \subset V$ (where $\mathbb{R}^{V_0}$ has the usual norm topology from the inner product). When V is finite, this is just the norm topology, but when V is infinite, it has strictly more open sets. Let $|K|$ be the subspace of $\mathbb{R}^V$ given by the union of the affine simplices defined by S, i.e., $t_0 v_0 + \cdots + t_n v_n \in |K|$ (for $v_0, \ldots, v_n \in V$) if and only if $\sum t_i = 1$, $t_i \geq 0$ (for all i), and $\{v_0, \ldots, v_n\} \in S$.

In this general context, the best way to define a possibly infinite triangulation (on a hausdorff space X) is as a homeomorphism from $|K| \to X$ for some possibly infinite simplicial complex K.

(a) Show that $U \subset |K|$ is open if and only if its intersection with every finite subcomplex is open. Show that $|K|$ is compact if and only if V is finite.

(b) Show that $\mathbb{R}$ and $\mathbb{R}^2$ are triangulable in this sense.

(c) Show by example that $|K|$ might not be locally compact.

(d) Show that $|K|$ is locally contractible.

$3^2\!/_3$. Some compact hausdorff spaces are not triangulable.

(a) Show that the Cantor set is not triangulable. (If you don't know the Cantor set, use the homeomorphic metric space X whose points are the infinite sequences $s = (s_1, s_2, \ldots)$ where $s_i \in \{0, 1\}$ and $d(s,t) = \sum |s_i - t_i| 2^{-i}$.)

(b) Show that the following compact subset of $\mathbb{R}^2$ is not triangulable: The union of the circle of radius $1/n$ around $(0, -1/n)$ for $n = 1, 2, 3, \ldots$.

$7^1\!/_2$. Let $K = (V, S)$ be a simplicial complex, and choose an order on its set of vertices. Define the homomorphism $sd_n \colon C_n(K) \to C_n(Sd\,K)$ by

$$sd_n(c_{a_0,a_1,\ldots,a_n}) = \sum_{\sigma} (-1)^{\sigma} c_{\{a_{\sigma 0}\},\{a_{\sigma 0},a_{\sigma 1}\},\ldots,\{a_{\sigma 0},\ldots,a_{\sigma n}\}}$$

where $\{a_0, \ldots, a_n\}$ is a simplex of K, the sum is over the set of permutations σ of $\{0, \ldots, n\}$, and $(-1)^{\sigma}$ denotes the sign of the permutation.

(a) Show that sd_* is a chain map, i.e., it commutes with the differential.

(b) Define $u \colon S \to V$ to be the function that takes an element A of S to the largest element in A (in the order chosen above). Show that u is a simplicial map and a simplicial approximation of the usual homeomorphism $|\,Sd\,K| \cong |K|$.

(c) Show that the composite

$$C_n(K) \xrightarrow{sd_n} C_n(Sd\,K) \xrightarrow{C_n(u)} C_n(K)$$

is the identity. (Next example sheet we'll show that u induces an isomorphism on homology; it then follows that sd_* induces an isomorphism on homology.)

End of Example Sheet 3s.

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