ALGEBRAIC GEOMETRY, SHEET III: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field.

0. (Pre-Question) In this course, an *algebraic variety* will be any Zariski open subset of a projective variety over k. This definition encompasses affine, quasi-affine, and projective varieties. They are known as *quasi-projective* varieties in the literature. The variety $\mathbb{A}^1 \times \mathbb{P}^1$ is a good example of an algebraic variety that is neither affine nor projective. Convince yourself that our definitions of rational functions, rational maps, and products work perfectly well in this setting.

Products and Graphs

1. Equip $\mathbb{P}^n \times \mathbb{P}^m$ with the following two topologies: (1) the subspace topology for the Zariski topology on $\mathbb{P}^{(n+1)(m+1)-1}$ under the Segre embedding; (2) the topology whose closed sets are finite intersections of vanishing loci of bihomogeneous[1] polynomials in variables
$$\{X_0, \ldots, X_n\} \cup \{Y_0, \ldots, Y_m\}$$
Prove that these topologies coincide.
2. Let $\varphi : X \to Y$ be a morphism of algebraic varieties. Prove that the graph Γ_φ is a closed subset of the product $X \times Y$.
3. Assume (the true fact[2]) that the morphism $\pi : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ given by projection onto the second factor is a closed map of topological spaces i.e. the image of every closed set is closed. Prove that if X is a projective variety and Y is any algebraic variety, the projection
$$X \times Y \to Y$$
is a closed map, and the image of a projective variety under a morphism is closed.

Morphisms

4. Let X be a projective variety. Using the previous question, prove that if X is irreducible and projective, any non-constant morphism $X \to \mathbb{P}^1$ is surjective. Deduce that any regular function (i.e. morphism to $\mathbb{A}^1$) on X is constant.
5. Describe all regular functions (i.e. morphisms to $\mathbb{A}^1$) on the variety $\mathbb{A}^2 \times \mathbb{P}^1$.
6. Consider the rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ given by projection from the point $[0 : 0 : 1]$. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the closure of the graph of
$$\varphi : \text{domain}(\varphi) \to \mathbb{P}^1.$$

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[1] Recall that this means that the polynomial is homogeneous in each set of variables separately, treating the other set as constant.

[2] This fact is hard to prove but worth trying.

Consider the projection $\pi : X \to \mathbb{P}^2$. Prove that $\pi^{-1}[0:0:1]$ is isomorphic to $\mathbb{P}^1$.

Local Rings and Tangent Spaces

7. For each integer $n \geq 1$, give an example of an algebraic variety X of dimension 1 such that there exists a point $p \in X$ with $\dim T_{X,p} = n$.
8. For each integer $k \geq 1$, give an example of a variety of dimension k, whose set of singular points has dimension $k-1$. Find an example of a variety whose set of singular points is itself a singular variety.
9. Let X be the affine variety consisting of the union of three lines passing through a point in $\mathbb{A}^2$. Let Y be the union of the three coordinate axes in $\mathbb{A}^3$. Prove that X and Y are not isomorphic.
10. Let $V \subset \mathbb{A}^n_k$ be an irreducible affine variety and let $p \in V$ be a point. Let $\mathfrak{m}_p$ denote the maximal ideal in the local ring of V at p. Prove that the dimension of the tangent space $T_{V,p}$ is equal to the dimension of $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Geometry of Curves

11. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation
$$y^2 = x(x-1)(x-\lambda)$$
for λ in k. Determine the values of λ for which E_λ is smooth. For E_λ smooth, prove that all morphisms $\mathbb{A}^1 \to E_\lambda$ are constant. ($\star\star$) For E_λ smooth, prove that all *rational maps* $\mathbb{A}^1 \dashrightarrow E_\lambda$ are constant.
12. Consider an affine plane curve $X = \mathbb{V}(f)$ for $f \in k[z_1, z_2]$ and let P be a smooth point on X. Show that $z_1 - z_1(P)$ is a local parameter[3] at P if and only if $(\partial f/\partial z_2)(P) \neq 0$.
13. Work in $\mathbb{P}^2_k$ over a field of characteristic not 2. Consider the curves $\mathbb{V}(Z_0^8 + Z_1^8 + Z_2^8)$ and $\mathbb{V}(Z_0^4 + Z_1^4 + Z_2^4)$ and determine whether they are smooth and/or irreducible.
14. Let $(\underline{b})$ be a vector of integers summing to 0. Construct a morphism $F : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following two conditions both hold: (1) for each negative entry b_i in $\underline{b}$, there exists a point p_i such that $F(p_i) = \infty$ in $\mathbb{P}^1$ and F ramifies at p_i to order $-b_i$, and (2) for each positive entry b_j, there is a point p_j such that $F(p_j) = 0$ in $\mathbb{P}^1$ and F ramifies at p_j to order b_j.

[3] Try to make contact between this local parameter and the intuitive description of "linear parts" of functions.