ALGEBRAIC GEOMETRY, SHEET II: LENT 2021

The symbol k will denote an algebraically closed field.

Homogeneous Coordinates and Projective Closure

1. A *line* in $\mathbb{P}^2$ is the vanishing locus of a homogeneous polynomial F of degree 1 in 3 variables. Observe that such a homogeneous polynomial also determines a linear subspace of k^3. Use this to prove that two distinct lines in $\mathbb{P}^2$ intersect at a single point.
2. *(Dual Projective Plane)* A line in $\mathbb{P}^2$ can be obtained by specifying 3 coefficients – namely those of X_0, X_1, and X_2 at least one of which is nonzero. When do two such specifications determine the same line? Deduce that the *set of all lines in* $\mathbb{P}^2$ is in natural bijection with $\mathbb{P}^2$.
3. Write down the projective closures of the following affine plane curves and calculate their intersections with the line at infinity. Plot the first two on a computer[1].
   - $xy = x^6 + y^6$.
   - $x^3 = y^2 + x^4 + y^4$
   - $y^2 = f(x)$ with $f(x)$ a polynomial of degree d.
4. Let V° be an affine variety in $\mathbb{A}^n$. Identify $\mathbb{A}^n$ with the subset of $\mathbb{P}^n$ where the first homogeneous coordinate is nonzero. Prove that if V° is irreducible then the projective closure of V° in $\mathbb{P}^n$ is also irreducible.
5. Consider the subset $V = \{(t, t^2, t^3) : t \in k\} \subset \mathbb{A}^3$. Observe that V is the vanishing locus of $y_2 - y_1^2$ and $y_3 - y_1^3$. Prove that this affine variety is irreducible. Show that the vanishing locus in $\mathbb{P}^3$ of $X_2X_0 - X_1^2$ and $X_0^2X_3 - X_1^3$ is not irreducible. Calculate generators for the ideal of the projective closure of V.

Some Projective Hypersurfaces

6. Prove that the conic $\mathbb{V}(X_0X_1 - X_2^2)$ in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. Deduce that the field of rational functions of this conic is $k(t)$.
7. The *Segre surface* $\Sigma_{1,1} \subset \mathbb{P}^3$ is given by $\mathbb{V}(Z_0Z_3 - Z_1Z_2)$. Calculate the field of rational functions of $\Sigma_{1,1}$. Describe the set of all lines contained on this surface. Plot an affine patch of this surface on a computer.
8. Construct two non-isomorphic irreducible cubic plane curves C_1 and C_2 in $\mathbb{P}^2$, such that the fields of rational functions of C_1 and C_2 are both isomorphic to $k(t)$.
9. Consider the *cubic surface* $S \subset \mathbb{P}^3$ given by $\mathbb{V}(Z_0^3 - Z_1^3 + Z_2^3 - Z_3^3)$. Find a line ℓ contained on this surface[2]. Choose any a plane containing ℓ and describe the irreducible

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[1]If you have a computer made by Apple you can do this on "Grapher". If not, Google will enumerate a large number of possibilities if you search for 2D and 3D graphers.

[2]This is part of a famous geometry. A (smooth) cubic surface contains exactly 27 lines no matter what the equation is. How many lines can you find?

components of its intersection with S. Plot an affine patch of this surface and your chosen line on a computer.

10. Let $X = \mathbb{V}(F)$ be a hypersurface in $\mathbb{P}^n$ and let ℓ be a line, i.e. a subvariety defined by $n-1$ homogeneous linear equations whose coefficient vectors are linearly independent. Show that X has a nonempty intersection with ℓ. Use this to prove that any hyperplane (i.e. the vanishing of a linear homogeneous polynomial) intersects X nontrivially.

Rational maps and Morphisms

11. The *Cremona transformation* is the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ sending[3] $[X_0 : X_1 : X_2]$ to $[\frac{1}{X_0} : \frac{1}{X_1} : \frac{1}{X_2}]$. Let ℓ be the line $\mathbb{V}(X_0 + X_1 + X_2)$ and let $U \subset \mathbb{P}^2$ be a nonempty open set where the map is defined. Calculate ideal of the Zariski closure of $\varphi(U \cap \ell)$.
12. Fix an integer $p > 0$ and consider the map $F_p : \mathbb{P}^n \to \mathbb{P}^n$ sending $[X_0 : \cdots : X_n]$ to $[X_0^p : \cdots : X_n^p]$. Prove that this map is defined (i.e. regular) everywhere and is therefore a morphism. Let ℓ denote the line in $\mathbb{P}^2$ given by $X_0 = X_1$. Calculate the homogeneous ideal associated to $F_p^{-1}(\ell)$.
13. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ sending (x, y) to (x, xy). Describe the image of this morphism. Calculate its Zariski closure.
14. *(Veronese maps)* Let $\{F_I\}$ be the set of degree d monomials in $n+1$ variables $Z_0, \ldots, Z_n$. Consider the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

sending a tuple $[Z_0 : \cdots : Z_n]$ to $[\cdots : F_I : \cdots]$, i.e. to the tuple of monomials of degree d. Check this map is defined (i.e. regular) everywhere. Find generators for the image of ν_d and prove that ν_d is an isomorphism onto its image[4].

Generalizations of $\mathbb{P}^n$. These final two introduce generalizations of projective space. The latter of these is a difficult question, but the example is important throughout geometry. Even if you do not solve this question, you may want to try to engage with it!

15. *(Weighted projective space)* Let $\underline{w} = (w_0, \ldots, w_n)$ be a tuple of positive integers. The *weighted projective space* $\mathbb{P}(\underline{w})$ is defined by

$$\mathbb{P}(\underline{w}) := \frac{k^{n+1} \setminus \{(0, \ldots, 0)\}}{\sim}$$

where $\sim$ is the relation that declares $(a_0, \ldots, a_n) \sim (\lambda^{w_0} a_0, \ldots, \lambda^{w_n} a_n)$ for any scalar $\lambda \in k^\star$. In analogy with $\mathbb{P}^n$, define homogeneous coordinates on $\mathbb{P}(\underline{w})$ by the coordinates on k^{n+1}. Let X_0, X_1, X_2 be such coordinates on $\mathbb{P}(1, 1, 2)$. Prove that the map

$$\mathbb{P}(1,1,2) \to \mathbb{P}^3 \quad ; \quad [X_0 : X_1 : X_2] \mapsto [X_0^2 : X_1^2 : X_0 X_1 : X_2]$$

[3]Perhaps more legally, by sending $[X_0 : X_1 : X_2]$ to $[X_1 X_2 : X_1 X_3 : X_2 X_1]$.

[4]This involves a lot of bookkeeping. If you find this too much, do the cases where $d = 3$, $n = 1$ and $d = 2$ and $n = 2$

is well-defined. Prove the image is Zariski closed and calculate the homogeneous ideal of the image.

16. *(Grassmannian)* An important generalization of projective space is called the Grassmannian. Let V be an n-dimensional vector space and $0 \leq k \leq n$ an integer. Let $G(k, V)$ be the set of k-dimensional linear subspaces of V.
    (a) Consider k linearly independent vectors $v_1, \ldots, v_k$ in V and choose a basis to represent them as a $k \times n$ matrix M. Observe that $GL(k)$ acts on the set of such matrices by left multiplication without affecting the associated vector subspace. Prove that the $k \times k$ minors of such a matrix give rise to a well-defined map
    $$\iota : G(k, V) \to \mathbb{P}^{\binom{n}{k}-1}.$$
    (b) Prove that ι is injective.
    (c) ($\star\star$) Prove that the image of ι is Zariski closed. (Hint: Given a subspace W represented by a matrix M_W, you may assume that the first $k \times k$ block of M_W is the identity. The rest of M_W is a $k \times (n-k)$ matrix A. How are the maximal minors of M_W related to the minors of A? The minors of A satsify relations coming from Laplace expansion. This gives you equations on an affine patch.)