ALGEBRAIC GEOMETRY, SHEET I: LENT 2021

Throughout this sheet, the symbol k will denote an algebraically closed field.

Topological Spaces of Varieties

1. The Zariski topology on $\mathbb{A}^n_k$ is defined by specifying its *closed sets* – these are the sets $\mathbb{V}(I)$ where I is an ideal in the polynomial ring. The open sets are precisely complements of the closed sets. Prove that these data satisfy the axioms of a topology on $\mathbb{A}^n_k$. Describe all the open sets in the topological subspace $\mathbb{V}(XY) \subset \mathbb{A}^2_k$.
2. We can equip $\mathbb{A}^2_k$ with the following two topologies. (1) The *Zariski topology* as in the previous question. (2) The *Z-product* topology, obtained by putting the Zariski topology on $\mathbb{A}^1_k$ and then viewing $\mathbb{A}^2_k$ as a product of topological spaces. Prove that the two resulting topological spaces are not homeomorphic.
3. **Topology Fact:** A topological space X is Hausdorff if and only if the diagonal subspace in $X \times X$ is closed, where $X \times X$ is given the product topology.
   Give $\mathbb{A}^1_k$ the Zariski topology and show that it is not Hausdorff. Deduce that the diagonal copy of $\mathbb{A}^1_k$ in $\mathbb{A}^1_k \times \mathbb{A}^1_k$ cannot be closed (note: the latter has the Z-product topology). Prove that, in contrast, the diagonal $\mathbb{A}^n_k$ inside $\mathbb{A}^{2n}_k$ is closed when the latter is equipped with the *Zariski* topology. [1]
4. Prove that $\mathbb{A}^2_k$ is compact when equipped with the Zariski topology. (If you have trouble with this, start with $\mathbb{A}^1_k$).
   Let $\mathbb{A}^3_k$ be affine 3-space with coordinates X, Y, t and consider the morphism $\pi : \mathbb{A}^3_k \to \mathbb{A}^1_k$ given by projecting onto the t coordinate. The preimage of any point in $\mathbb{A}^1_k$ can be viewed as a copy of $\mathbb{A}^2_k$. Construct a variety $X \subset \mathbb{A}^3_k$, such that for nonzero t, the set $\pi^{-1}(t)$ is a union of two intersecting lines whereas $\pi^{-1}(0)$ is a union of two parallel lines.[2]

Irreducible Components

5. Consider the ideal $(X^2 + Y^2 + Z^2, X^2 - Y^2 - Z^2 + 1)$ in $k[X, Y, Z]$. Calculate the irreducible components of the vanishing locus of this ideal.
6. Let $V \subset \mathbb{A}^n_k$ be an affine variety. Suppose $V = V_1 \cup \cdots \cup V_n$ and $V = V'_1 \cup \cdots \cup V'_m$ are two decompositions into irreducible components. Assume that no V_i is contained in V_j for $i \neq j$ and similarly for the V'_i – i.e. the decompositions are non-redundant. Prove

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[1] The conclusion is that *from the point of view of ring theory* affine space behaves as if it were Hausdorff, even though it isn't!

[2] The conclusion this time is that although $\mathbb{A}^2_k$ is compact from the point of view of the Zariski topology, there is a limit (i.e. the intersection point of the lines) that doesn't exist. Therefore *from the point of view of ring theory* $\mathbb{A}^2_k$ behaves as if it weren't compact.

that $n = m$ and the two decompositions coincide up to reordering. (Hint: Make each V_i interact with all the V'_j and vice versa.)

7. Prove that a prime ideal is equal to its own radical. Give an example of a radical ideal that is not prime. Calculate the radical of the ideal (XY, X^2) in $k[X, Y]$.
8. Assume that the characteristic of k is not 2 or 3. Show that the polynomial $Y^2 - X^3 - X$ is irreducible in $k[X, Y]$. Deduce that the vanishing locus of this polynomial is irreducible. Make note of the property of $k[X, Y]$ that you are using.

Examples and Counterexamples

9. Prove that the affine curve given by $XY = 1$ in $\mathbb{A}^2_k$ is not isomorphic to $\mathbb{A}^1_k$. Prove that the affine curve given by $Y = X^2$ is isomorphic to $\mathbb{A}^1_k$.
10. Let C be the affine variety $\mathbb{V}(Y^2 - X^3)$ in $\mathbb{A}^2_k$ and D be the affine variety $\mathbb{V}(ZW - 1)$ in $\mathbb{A}^2_k$. Find all morphisms from C to D. For both varieties, calculate the field of fractions of the coordinate ring (also known as the field of rational functions).
11. Give an example of a non-algebraically closed field field K and I a proper ideal in $K[\underline{X}]$ such that $\mathbb{V}(I) \subset \mathbb{A}^n_K$ is empty.
12. **Twisted Affine Curves.** Let $V \subset \mathbb{A}^3_k$ be the set $\{(t, t^2, t^3) : t \in k\}$. Calculate $I(V)$. Prove that the quotient ring $k[X, Y, Z]/I(V)$ is isomorphic to a polynomial ring. Calculate the ideal of $W = \{(t^3, t^4, t^5) : t \in k\}$. (A difficult calculation reveals that $I(W)$ cannot be generated by fewer than 3 generators.)

Products

13. Let V and W be affine varieties in $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$ respectively. Prove that the product $V \times W \subset \mathbb{A}^{n+m}$ is also an affine variety. Describe the coordinate ring of $V \times W$ in terms of the coordinate rings of those of V and W. Describe a variety whose coordinate ring is isomorphic to the product ring $k[X] \times k$.
14. **Difficult But Worth Knowing.** Prove that if V and W are *irreducible* affine varieties in $\mathbb{A}^n_k$ and $\mathbb{A}^m_k$ then $V \times W$ is also irreducible in $\mathbb{A}^{n+m}_k$.

Zooming in to the Singularity

15. Consider the polynomial $Y^2 - X^2(X + 1)$ and sketch the real points of the associated variety $C(\mathbb{R}) \subset \mathbb{R}^2$. Let $\mathbb{D}_\epsilon$ be an open ball around $(0, 0)$ in $\mathbb{R}^2$ in the standard Euclidean topology of some small radius ϵ. Observe that $\mathbb{D}_\epsilon \cap C(\mathbb{R})$ is homeomorphic to a union of two axes in $\mathbb{R}^2$. We will now see a shadow of this algebraically.
    Consider the ring $k[\![X, Y]\!]$ of formal power series in two variables. Prove that there exists an element $F(X, Y)$ in this ring such that $F(X, Y)^2 = (1 + X)$. Deduce that the element $Y^2 - X^2(X + 1)$ can be factorised as:
    $$Y^2 - X^2(X + 1) = (Y - XF)(Y + XF).$$
    Note that F is an invertible element in this ring and meditate on the relationship between the two parts of this problem.