ALGEBRAIC GEOMETRY, SUPPLEMENTARY EXAMPLES: LENT 2020

Throughout this sheet, the symbol k will denote an algebraically closed field.

The following sheet contains a number of calculations and examples that we won't have time to cover in the course. These are meant to build your intuition for the techniques and objects that we've been developing.

1. Given three mutually disjoint lines in $\mathbb{P}^3$, prove that there is a unique quadric surface containing these three lines.
2. How many lines in $\mathbb{P}^3$ intersect four given disjoint lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$?
3. Given a homogeneous ideal I, its vanishing locus $X = \mathbb{V}(I) \subset \mathbb{P}^n_k$ is a projective variety. The *affine cone over* X is the vanishing locus of the same equations, but in $\mathbb{A}^n_k$. Prove that the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$ in its Segre embedding inside $\mathbb{P}^3$ is not smooth. Classify all projective varieties X whose affine cone is smooth.
4. A morphism $\varphi : X \to Y$ is said to *separate points* if it is one-to-one. It *separates tangent vectors at* $p \in X$ if the induced map $d\varphi_p$ from the tangent space $T_{X,p} \to T_{Y,\varphi(p)}$ is injective. Which of the following morphisms separates points resp. tangent vectors? (Assume we're working over a characteristic 0 field).
   (a) $\mathbb{A}^1 \to \mathbb{A}^2$ sending $t \mapsto (t, t^2)$.
   (b) $\mathbb{A}^1 \to \mathbb{A}^2$ sending $t \mapsto (t^2, t^3)$.
   (c) $\mathbb{A}^1 \mapsto \mathbb{A}^2$ sending $t \mapsto (t^2 - 1, t(t^2 - 1))$.
   Note from this example that a bijective morphism need not be an isomorphism!
5. Let X be an affine variety with coordinate ring $k[X]$. Let G be a finite group acting on $k[X]$ (equivalently on X) with the action given by a homomorphism
$$G \to \mathrm{Aut}(k[X])$$
   where the latter is the group of ring automorphisms. Construct an algebraic variety with underlying set equal to X/G, such that the quotient
$$X \to X/G$$
   is a morphism[1]. (Hint: What are the natural functions on X/G? Also use Question 14, Sheet I)[2].

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[1]The situation is better – the quotient of a quasi-projective variety by a finite group always exists.

[2]If G is taken to be a finite cyclic subgroup of $GL(2, \mathbb{C})$, the resulting quotients of $\mathbb{A}^2_{\mathbb{C}}$ are a very interesting family of singular varieties. Try seeing why and where they are singular.

6. Consider the *nodal cubic* $C \subset \mathbb{A}^2$ given by the vanishing of $f = y^2 - x^2(x+1)$. Prove that it has a singular point at the origin. Draw this curve over the real numbers. Describe the Euclidean topological space associated to C over the complex numbers.
7. We continue with the equation from the previous question. The coordinate ring of the nodal cubic is $k[x, y]/(f)$. If we replaced $k[x, y]$ by the *ring of formal power series in two variables*[3], we could instead consider the quotient
$$k[\![x, y]\!]/(f).$$
Prove that this ring is isomorphic to the ring
$$k[\![u, v]\!]/(uv).$$
Examining your picture at $(0, 0)$, what can you deduce about the information contained in this "power series version" of the coordinate ring?

[3] This is the set of power series with non-negative exponents in x and y, coefficients in k, and no convergence condition.