ALGEBRAIC GEOMETRY, SHEET IV: LENT 2020

Throughout this sheet, the symbol k will denote an algebraically closed field. Curves will be smooth, projective, and irreducible unless explicitly stated otherwise.

1. Prove that if X and Y are smooth projective curves then $k(X) \cong k(Y)$ if and only if $X = Y$.
2. Let X be a smooth degree d plane curve. Construct a morphism from X to $\mathbb{P}^1$ of degree $d-1$.
3. Let X be a curve and $p \in X$. Prove that there exists a non-constant rational function on X that is regular away from p.
4. Let $f(x)$ be a cubic polynomial with distinct roots and assume that k has characteristic different from 2. Let X be projective closure of the affine curve $y^2 - f(x) = 0$. Consider the differential $\omega = \frac{dx}{y}$ and prove that $[\mathrm{div}(\omega)] = 0$. Deduce from this that the genus of X is 1.
5. Let X be the projective closure of the affine curve $y^3 = x^4 + 1$. Prove that this curve is smooth and prove that it has a unique point at infinity. Calculate the zeroes and poles of the differential $\frac{dx}{y^2}$.
6. *(The abstract algebraic structure on products)* Let $\mathbf{Z} = \{Z_0 \ldots, Z_k\}$ and $\mathbf{W} = \{W_0, \ldots, W_\ell\}$ be two sets of variables. A bihomogeneous polynomial is a polynomial that is homogeneous separately in $\mathbf{Z}$ and $\mathbf{W}$ but possibly of different degrees. The vanishing loci of bihomogeneous polynomials in $\mathbb{P}^k \times \mathbb{P}^\ell$ define the closed sets in a topology. Prove that these closed sets coincide with the closed sets of $\mathbb{P}^k \times \mathbb{P}^\ell$ obtained via the Segre embedding. If you are confused about the Segre embedding, first solve the question with $k = \ell = 1$.
7. *(Canonical class for curves on a quadric)* Let F be a bihomogeneous polynomial of bidegree (d_1, d_2) in two sets of two variables $\mathbf{Z}$ and $\mathbf{W}$. Let
$$X = \mathbb{V}(F) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$
Let H_1 be the divisor on X associated to the curve $L_1(Z_0, Z_1) = 0$ where L_1 is a linear form in two variables. Similarly, define H_2 to be the divisor associated to a linear form $L_2(W_0, W_1)$ [1]. By adapting the proof given for $\mathbb{P}^2$ in lecture, show that
$$K_X = (d_1 - 2)H_1 + (d_2 - 2)H_2.$$
8. (*Degree-genus on a quadric surface*) Prove that a smooth projective irreducible curve of bidegree (d_1, d_2) on $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $(d_1 - 1)(d_2 - 1)$. Deduce that a smooth quadric

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[1] These are preimages of points in $\mathbb{P}^1$ under the two different projections, taken with the appropriate multiplcities.

surface in $\mathbb{P}^3$ (and hence $\mathbb{P}^3$ itself) contains a smooth projective curve of every genus. Observe that $\mathbb{P}^2$ does not contain a smooth curve of genus 2.

9. A curve is said to be *hyperelliptic* if it has genus at least 2 and admits a non-constant 2-to-1 morphism to $\mathbb{P}^1$. Using Riemann–Roch, prove that every curve of genus 2 is hyperelliptic. Prove that there exists a hyperelliptic curve of genus g for any $g \geq 2$.
10. Give an example of a quartic plane curve $X \subset \mathbb{P}^2$ and a line $\ell \subset \mathbb{P}^2$ such that ℓ is tangent to X at two distinct points. What does this tell you about the divisor on X associated to ℓ?
11. Construct a smooth projective variety of dimension 2, i.e. an algebraic surface, that does not contain a smooth curve of genus 0. For every non-negative h, can you construct a surface that contains no smooth curves of genus h?

A couple of thoughts on the ambient mathematics. The canonical class formula for curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are instances of a fundamental concept, known as *adjunction*. The theory of divisors can be generalized to higher dimensional algebraic varieties, where rather than linear combinations of points, one considers linear combinations of codimension 1 subvarieties (rational functions tend to have poles along codimension 1 subvarieties.) Our study of divisors and differentials did not use the dimension 1 hypothesis in a serious way, and one can define the canonical divisor of any non-singular algebraic variety in this way. Adjunction relates the canonical class on Y to the canonical class on a smooth subvariety $X \subset Y$.

Although there exists a hyperelliptic curve of every genus, once $g \geq 3$, "most" curves are non-hyperelliptic. To make this precise, one needs an appropriate parameter space $\mathcal{M}_g$ for all smooth curves of genus g. This space turns out to be an algebraic variety of dimension $3g-3$ and points of this space correspond to genus g curves. The hyperelliptic curves inside $\mathcal{M}_g$ form a subvariety of dimension $2g-1$. Once $g \geq 3$, this is a proper subvariety in $\mathcal{M}_g$.

As we've seen (or will see, depending on when you read this) the condition of X being non-hyperelliptic endows X with some nice structure – it can be embedded into projective space by its canonical class. An elliptic curve can always be described as a plane cubic, a genus 2 curve is a curve of degree $(3,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, a non-hyperelliptic genus 3 curve can always be expressed as a plane quartic, while a non-hyperelliptic genus 4 curve can always be described as a $(3,3)$ curve on $\mathbb{P}^1 \times \mathbb{P}^1$ or equivalently, an intersection of a quadric and a cubic in $\mathbb{P}^3$. The coefficients in the equations give a "parameteriziation" of $\mathcal{M}_g$ for small g, by an affine space. Once g gets large there is *provably* no way to "parameterize" an open set in the variety of all genus g curves – this is a celebrated result!

The lines in question 9 that are twice tangent to a quartic are very sensibly called *bitangents*. Any smooth plane curve has finitely many bitangent lines. For plane quartics this number is 28. We've also seen that cubic surfaces contain 27 lines. Remarkably, these two "'enumerative" geometric questions turn out to be related by geometric transformstions, most clearly seen via blowups and the theory of del Pezzo surfaces.