ALGEBRAIC GEOMETRY, SHEET III: LENT 2020

Throughout this sheet, the symbol k will denote an algebraically closed field.

1. Let $\varphi : X \to Y$ be a morphism of algebraic varieties. Prove that the graph Γ_φ is a closed subset of the product $X \times Y$.
2. Assume (the true fact[1] that) the morphism $\pi : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ given by projection onto the second factor is a closed map of topological spaces. Prove that for X projective and Y quasi-projective the projection
$$X \times Y \to Y$$
is also a closed mapping. Deduce that the image of a projective variety is closed.
3. Using the ideas above prove that the only regular functions on a projective variety are constant functions.
4. Calculate the ring of all regular functions (i.e. everywhere defined morphisms $X \to \mathbb{A}^1$) on the quasi-projective variety $X = \mathbb{A}^2 \times \mathbb{P}^1$.
5. Let $\mathcal{A}$ be a union of hyperplanes in $\mathbb{P}^n$ be a collection of $k > n$ hyperplanes in general position. Exhibit the complement $\mathbb{P}^n \setminus \mathcal{A}$ as a closed subvariety an algebraic torus, i.e. a product of copies of $\mathbb{G}_m = (k^\star)$. (Start with $n = 1$ and $|\mathcal{A}| = 3$.)
6. Suppose that the characteristic of k is not equal to 2. Show that any irreducible smooth quadric in $\mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$. If $n = 3$, prove that every such quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.
7. Recall the blowup of $\mathbb{A}^2$ at the origin, defined in lecture[2]: $\pi : X \to \mathbb{A}^2$. The *strict transform* of a subvariety $Y \subset \mathbb{A}^2$ is defined as the closure of $\pi^{-1}(Y \setminus (0, 0))$ inside X. Calculate the strict transform of the curve given by
$$Y = \mathbb{V}(y^2 - x^2(x + 1)).$$
To see the geometry here, ask the internet to show you a graph the real points of this algebraic variety and remember that the blowup separated lines through the origin based on slope[3].
8. Let d be a positive integer and $(a_1, \ldots, a_n)$ be positive integers summing up to d. Construct a morphism of smooth curves $X \to \mathbb{P}^1$ such that the preimage of ∞ consists of exactly n points, with ramification indices given by $a_1, \ldots, a_n$. Let (b_i) be a vector of integers summing to 0. Construct a morphism $X \to \mathbb{P}^1$ having poles with orders given by the negative entries in b_i and zeros orders given by the positive ones.

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[1]This fact is hard to prove but worth trying.

[2]If you don't have access to notes from the lecture, you can consult my colleague, Dr. Wikip Edia.

[3]This blowup has been lurking in the background. If you take 6 generically chosen points in $\mathbb{P}^2$ and blow them up you get an algebraic surface. One can show that every cubic surface has this form!

(Important lesson: It is easy to construct rational functions on $\mathbb{P}^1$ with prescribed zeroes and poles.)

9. Consder an affine plane curve $X = \mathbb{V}(f)$ for $f \in k[z_1, z_2]$ and let P be a smooth point. Show that $z_1 - z_1(P)$ is a local parameter at a smooth point if and only if $\partial f/\partial z_2(P) \neq 0$.
10. Let p be a smooth point on an irreducible curve X and let π be a local parameter. Prove that the dimension of $\mathcal{O}_{X,p}/\mathfrak{m}_p^n$ is equal to n for every positive integer n. Work this out explicitly when $X = \mathbb{A}^1$ and $p = 0$ and interpret the general case based on this.
11. We work in $\mathbb{P}^2_k$ over a field of characteristic different from 2. Let $X_1 = \mathbb{V}(Z_0^8 + Z_1^8 + Z_2^8)$ and $X_2 = \mathbb{V}(Z_0^4 + Z_1^4 + Z_2^4)$ be curves. Show that each is smooth and irreducible. Let $\varphi : X_1 \to X_2$ be the morphism sending Z_i to Z_i^2. Determine the degree of this morphism and compute the ramification indices for all points of X_1.
12. *(Elliptic curves are not rational* Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation
$$y^2 = x(x-1)(x-\lambda).$$
We will show that E_λ admits no non-constant rational maps $F : \mathbb{A}^1 \dashrightarrow E_\lambda$.
    (a) Write
    $$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)}\right)$$
    where the numerator-denominator pairs have no common factors. Conclude that
    $$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}.$$
    is an equality of fractions that cannot be simplified any further. Analyze the factorization into (linear) factors of both numerators and denominators. Conclude that $f, g, f-g, f-\lambda g$ must be perfect squares.
    (b) Prove the following: If f, g are polynomials in $k[t]$ such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f-g, f-\lambda g$ are perfect squares, then f and g are constant. Use this to deduce that any rational map to E_λ from $\mathbb{P}^1$ is constant.

    Note that when $\lambda = 1$ the associated curve becomes birational to $\mathbb{P}^1$!
13. $(\star)$ *(A general quartic contains no lines)* Let $\mathbb{G} = G(2, 4)$ be the Grassmannian of linear 2-planes in k^4, which you may assume is an algebraic variety. This is naturally identified with the set of lines in $\mathbb{P}^3$. Let $\mathbb{P}$ be the projective space of all quartic hypersurfaces, i.e. of nonzero homogeneous degree 4 polynomials in 4 variables, up to scalar.
    (a) Find a quartic homogeneous polynomial in 4 variables whose associated hypersurface contains infinitely many lines.
    (b) Let $\mathcal{Y}$ be the subset of $\mathbb{P} \times \mathbb{G}$ given by
    $$\mathcal{Y} = \{(S, [\ell]) \in \mathbb{P} \times \mathbb{G} : \text{the line } \ell \text{ is contained in the surface } S\}$$
    Prove that $\mathcal{Y}$ is a projective variety[4].

[4]Hint: To do this, use the fact from the previous sheet that $\mathbb{G}$ embeds in a projective space via the minors of a matrix. This embeds $\mathcal{Y}$ into a product of projective spaces. In the coordinates on this space, what is the condition for ℓ to lie on S?

(c) Consider the projections $p : \mathcal{Y} \to \mathbb{P}$ and $q : \mathcal{Y} \to \mathbb{G}$. Convince yourself that these are morphisms of algebraic varieties. Calculate the dimension of $q^{-1}([\ell])$.
(d) Calculate the dimension of $\mathbb{G}$ and use this to determine the dimension of $\mathcal{Y}$.
(e) Deduce that a generically chosen smooth quartic surface in $\mathbb{P}^3$ contains no lines.

Notice the pattern: a plane in $\mathbb{P}^3$ has huge numbers of lines. A smooth quadric surface (i.e. $\mathbb{P}^1 \times \mathbb{P}^1$) has two families of lines, or two "rulings". A cubic surface has finitely many lines. A general quartic surface contains no lines.