Throughout this sheet, the symbol \( k \) will denote an algebraically closed field.

1. Let \( X \subset \mathbb{P}^2 \) be a projective variety. A morphism \( \mathbb{P}^1 \to X \) is a polynomial map
   \[
   \varphi([X_0 : X_1]) = (f_0([X_0 : X_1]), f_1([X_0 : X_1]), f_2([X_0 : X_1]))
   \]
   where \( f_i \) are homogeneous of the same degree, such that \( f(\mathbb{P}^1) \subset X \). For any irreducible conic \( C \subset \mathbb{P}^2 \), there is a bijective morphism
   \[
   \varphi : \mathbb{P}^1 \to C.
   \]

2. Let \( X = \mathbb{V}(F) \) be a hypersurface in \( \mathbb{P}^n \) and let \( \ell \) be a line, i.e. a subvariety defined by \( n - 1 \) linearly independent linear homogeneous polynomials. Show that \( X \) intersects \( \ell \) in a nonempty set.

3. A quasi-projective variety \( X \) (or simply a variety in this course) is a complement \( Y \setminus W \) where \( Y \) is a projective variety and \( W \subset Y \) is a (possibly empty) closed subvariety. Give examples of quasi-projective varieties that are neither affine nor projective. Prove that any quasi-projective variety \( X \) is covered by affine varieties.

4. Consider the morphism \( \mathbb{A}^2 \to \mathbb{A}^2 \) sending \( (x,y) \) to \( (x,xy) \). Prove that the image of this morphism is not quasi-projective.

5. Let \( I \subset k[z_1, \ldots, z_n] \) be an ideal. Let \( f_1, \ldots, f_r \) be generators for this ideal. Prove or give a counterexample to the following statement relating homogenizations of polynomials and ideals:
   \[
   I^h = \langle f_1^h, \ldots, f_r^h \rangle \subset k[Z_0, \ldots, Z_n].
   \]

6. A collection of points in \( \mathbb{P}^n \) is said to be in general position if no subset of \( n + 1 \) or fewer points is linear dependent (as vectors in \( k^{n+1} \)). If \( S \) is a collection of \( 2n \) points in \( \mathbb{P}^n \), prove that \( S \) is the zero locus of a collection of quadratic polynomials.

7. Recall that the Segre surface \( \Sigma_{1,1} \subset \mathbb{P}^3 \) is given by \( \mathbb{V}(Z_0Z_3 - Z_1Z_2) \). Calculate the field of rational functions of \( \Sigma_{1,1} \). Describe the set of all lines contained on this surface.

8. Consider the cubic surface \( S \subset \mathbb{P}^3 \) given by \( \mathbb{V}(Z_0^3 + \ldots + Z_3^3) \). Find a line \( \ell \) contained on this surface\(^1\). Consider the collection of planes in \( \mathbb{P}^3 \) passing through this \( \ell \) and describe the corresponding collection of equations for these planes. Given such a plane \( H \) containing \( \ell \), describe the curve \( H \cap S \). Given a general plane in \( \mathbb{P}^3 \) (in particular, not necessarily containing \( \ell \)) describe its intersection with \( S \).

9. Construct two non-isomorphic irreducible cubic plane curves \( C_1 \) and \( C_2 \) in \( \mathbb{P}^2 \), such that the fraction field of the coordinate rings of \( C_1 \) and \( C_2 \) are both isomorphic to \( k(z) \). Draw appropriate pictures.

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\(^1\)This is part of a famous geometry. A (smooth) cubic surface contains exactly 27 lines no matter what the equation is. Can you find the 27 lines in this case?
10. (Veronese varieties) Let \( \{F_i\} \) be the set of degree \( d \) monomials in \( n + 1 \) variables \( Z_0, \ldots, Z_n \). Consider the map
\[
\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}
\]
sending a tuple \([Z_0 : \cdots : Z_n]\) to \([\cdots : F_d : \cdots]\), i.e. to the tuple of monomials of degree \( d \). Find generators for the image of \( \nu_d \) and prove that \( \nu_d \) is an isomorphism onto its image.

11. Write down the projective closures of the following curves, determine the points at infinity, and find all singular points:
- \( xy = x^6 + y^6 \)
- \( x^3 = y^2 + x^4 + y^4 \)
- \( y^2 = f(x) \) with \( f(x) \) a polynomial of degree \( d \).

12. (Dual projective space) Let \( V \) be a finite dimensional vector space. Prove that the set of hyperplanes in \( \mathbb{P}(V) \) is naturally isomorphic to \( \mathbb{P}(V^*) \), where \( V^* \) is the dual vector space. Using the fact that two distinct lines intersect at a unique point in \( \mathbb{P}^2 \), deduce the fact that there is a unique line through two distinct points in \( \mathbb{P}^2 \).

13. (Grassmannian) In this exercise, you will contemplate an important generalization of projective space known as the Grassmannian. Let \( V \) be an \( n \)-dimensional vector space and \( 0 \leq k \leq n \) an integer. Let \( G(k, V) \) be the set of \( k \)-dimensional linear subspaces of \( V \).
   (a) Consider \( k \) linearly independent vectors \( v_1, \ldots, v_k \) in \( V \) and choose a basis to represent them as a \( k \times n \) matrix \( M \). Observe that \( GL(k) \) acts on the set of such matrices by left multiplication without affecting the associated vector space. Prove that the \( k \times k \) minors of such a matrix give rise to a well-defined map
\[
\iota : G(k, V) \to \mathbb{P}^{\binom{n}{k}-1}.
\]
   (b) Prove that \( \iota \) is injective.
   (c) \( \star \) Prove that the image of \( \iota \) is Zariski closed. (Hint: Given a subspace \( W \) represented by a matrix \( M_W \), you may assume that the first \( k \times k \) block of \( M_W \) is the identity. The rest of \( M_W \) is a \( k \times (n-k) \) matrix \( A \). How are the maximal minors of \( M_W \) related to the minors of \( A \)? The minors of \( A \) satisfy relations coming from Laplace expansion. This gives you equations on an affine patch.)

14. Consider \( C \) a smooth curve given by the projective closure of an affine curve of the form \( y^2 = f(x) \) over the complex numbers. Construct a morphism
\[
\pi : C \to \mathbb{P}^1
\]
such that at all but finitely many points \( q \) of \( \mathbb{P}^1 \), the preimage \( \pi^{-1}(q) \) consists of exactly two points\(^2\).

15. \( \star \) Let \( d \geq 1 \) be an integer and \( C \subset \mathbb{P}_C^2 \) the union of a collection of \( d \) lines. Assume the lines are chosen generically. In the Euclidean topology in \( \mathbb{P}^2 \), describe the topological space of \( C \) and draw a picture of it. Let \( f_d \) be a degree \( d \) homogeneous polynomial
\[\text{\textsuperscript{2}Given a morphism } C \to \mathbb{P}^1 \text{, the number of preimages of a generic point on } \mathbb{P}^1 \text{ is the degree of the map, which in this case will be 2.}\]
in 3 variables, and assume the coefficients of $f_d$ are chosen generically. Based on your analysis above, can you guess a description of the Euclidean topological space $V(f_d)$? It may help you to carefully compare $V(Z_0Z_1)$ and $V(Z_0Z_1 - Z_2^2)$. 