Part II Algebraic geometry

Example Sheet I, 2019

In all problems, you may assume k is algebraically closed. The main point of this example sheet is to play with some examples of algberaic varieties – and examples which we didn't have time to cover in lectures.

1. Let $X \subseteq \mathbf{A}^n$ be an arbitrary subset. Show that $Z(I(X))$ coincides with the closure of X, i.e., the smallest Zariski closed subset of $\mathbf{A}^n$ containing X.

2. Show that any non-empty open subset of an irreducible affine variety is dense and irreducible. Show that if an irreducible affine variety is Hausdorff, it consists of a single point.

3 i) A topological space is called *Noetherian* if it satisfies the descending chain condition for closed subsets. Show that affine varieties are Noetherian in the Zariski topology.

ii) Show that an affine algebraic variety X is a finite union of irreducible affine varieties.

iii) The irreducible varieties that occur in (ii) are well defined; they are called the *irreducible components* of X. Here is a precise statement: Suppose that $X = X_1 \cup \cdots \cup X_n$ and $X = X'_1 \cup \cdots \cup X'_m$ are two decompositions into irreducible components, such that $X_i \not\subseteq X_j$ for any $i \neq j$, and $X'_i \not\subseteq X'_j$ for any $i \neq j$. Show that $n = m$ and after reordering, $X_i = X'_i$.

4. Let $Y \subseteq \mathbf{A}^2$ be the curve given by $xy = 1$. Show that Y is not isomorphic to $\mathbf{A}^1$. Find all morphisms $\mathbf{A}^1 \to Y$ and $Y \to \mathbf{A}^1$.

5. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety, determine $I(Y)$, and show that $A(Y)$ is a polynomial ring in one variable. Y is called the *twisted cubic.*

6. Let $Y = Z(x^2 - yz, xz - x)$. Show that Y has 3 irreducible components. Describe them, and their corresponding prime ideals.

7. Show that if $X \subseteq \mathbf{A}^n$, $Y \subseteq \mathbf{A}^m$ are affine varieties, then $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^m = \mathbf{A}^{n+m}$ is a Zariski closed subset of $\mathbf{A}^{n+m}$. You might also try to show that if X and Y are irreducible, $X \times Y$ is irreducible (but this is hard!).

8. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t^3, t^4, t^5) \mid t \in k\}$. Show that Y is an affine variety, and determine $I(Y)$. Show $I(Y)$ cannot be generated by two elements.

9. Suppose the characteristic of k is not 2. Show that there are no non-constant morphisms from $\mathbf{A}^1$ to $E = Z(y^2 - x^3 + x) \subseteq \mathbf{A}^2$. [Hint: Consider the map $A(E) \to A(\mathbf{A}^1) = k[t]$, and the images of x and y under this map. Then use the fact that $k[t]$ is a UFD.]

10. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, and consider $Y = Z(yf - 1) \subseteq \mathbf{A}^{n+1}$, with coordinates $x_1, \ldots, x_n, y$. Show that Y is irreducible. Show that the projection $\mathbf{A}^{n+1} \to \mathbf{A}^n$ given by $(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n)$ induces a morphism $Y \to \mathbf{A}^n$ which is

a homeomorphism to its image $D(f) := \{(a_1, \ldots, a_n) \in \mathbf{A}^n \mid f(a_1, \ldots, a_n) \neq 0\}$. This gives the Zariski open set $D(f)$ the structure of an algebraic variety.

11. Let $f, g \in k[x, y]$ be polynomials, and suppose f and g have no common factor. Show there exists $u, v \in k[x, y]$ such that $uf + vg$ is a non-zero polynomial in $k[x]$.

Now let $f \in k[x, y]$ be irreducible. The variety $Z(f)$ is called an affine *plane curve*. Show that any proper subvariety of $Z(f)$ is finite.

12. Show that $G = GL_n(k)$ is an affine variety, and that the multiplication and inverse maps are morphisms of algebraic varieties. We say G is an *affine algebraic group*. Show that if G is an affine algebraic group, and H is a subgroup which is also a closed subvariety of G, then H is also an affine algebraic group.

Hence show $SL_n(k)$, $O_n(k) = \{A \mid AA^T = I\}$, and the group of invertible upper triangular matrices are also affine algebraic groups.

13. Let $Mat_{n,m}$ denote the set of n by m matrices with coefficients in k; this is an affine variety isomorphic to $\mathbf{A}^{nm}$.

i) Show that the set of 2 by 3 matrices of rank ≤ 1 is an affine variety.

ii) Show that the matrices of rank 2 in $Mat_{2,3}$ is a Zariski open subset. [Warning: It is not an affine variety, for the same reason $\mathbf{A}^2 \setminus \{(0, 0)\}$ is not.]

iii) Show that the set of matrices in $Mat_{n,m}$ of rank $\leq r$ is an affine subvariety.

14. Let $G = \mathbf{Z}/2$ act on $k[x, y]$ by sending $x \mapsto -x$, $y \mapsto -y$. Show the algebra of invariants $k[x, y]^G$ defines an affine subvariety X of $\mathbf{A}^3$ by explicitly computing it in terms of generators and relations. X is called the *rational doublepoint*.

What is the relation of the points of X to the orbits of G on $\mathbf{A}^2$?