Algebraic Geometry, Part II, Example Sheet 3,2018

Assume throughout that the base field k is algebraically closed. This example sheet is harder (and longer) than the previous ones, so don't despair if you don't get all the problems!

1. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $X_1X_2^2 - X_3^3 \in k[X_0, \dots, X_3]$. Find the dimension of the tangent space at all the singularities.

2. Let $\phi : X \to Y$ be a morphism of affine varieties.

   (i) Show that for all $p \in X$, there is a linear map
   $$d\phi : T_pX = Der(k[X], ev_p) \to T_{\phi(p)}Y = Der(k[Y], ev_{\phi(p)}).$$

   (ii) If ϕ is defined by an m-tuple of polynomials $(\Phi_1, \dots, \Phi_m) \in k[X]^m$, write $d\phi$ in terms of the Φ_i.

   (iii) Deduce from (i) that if $\phi : X \to Y$ is a morphism of varieties, there is a linear map $d\phi : T_pX \to T_{\phi(p)}Y$.

3. * In this question, we will show that 'the generic hypersurface is smooth' — that is, that the set of smooth hypersurfaces of degree d is dense in the variety of all hypersurfaces of degree d in $\mathbb{A}^n$

   Let $n, d \geq 1$, and let $X = \{f \in k[x_1, \dots, x_n] \mid \deg f \leq d\}$, and $Z = \{(f, p) \in X \times \mathbb{A}^n \mid f(p) = 0$ and $k[x_1, \dots, x_n]/(f)$ is *not* the ring of functions of an affine variety which is smooth at $p\}$.

   (This is somewhat clumsy phrasing!)

   i) Show $X \simeq \mathbb{A}^N$ for some N [you need not determine N] and that Z is a Zariski closed subvariety of $X \times \mathbb{A}^n$.

   ii) Show that the fibers of the projection map $Z \to \mathbb{A}^n$ are linear subspaces of dimension $N - (n+1)$.

   Conclude that $\dim Z = N - 1 < \dim X$.

   iii) Hence show that $\{f \in X \mid \deg f = d, Z(f) \text{ smooth }\}$ is dense in X.

   [Quote any theorems of lectures you need].

4. Let P be a smooth point of the irreducible curve V. Show that if $f, g \in k(V)$ then $v_P(f+g) \geq \min(v_P(f), v_P(g))$, with equality if $v_P(f) \neq v_P(g)$.

5. If P is a smooth point of an irreducible curve V and $t \in \mathcal{O}_{V,P}$ is a local parameter at P, show that $\dim_k \mathcal{O}_{V,P}/(t^n) = n$ for every $n \in \mathbb{N}$.

6. Show that $V = Z(X_0^8 + X_1^8 + X_2^8)$ and $W = Z(Y_0^4 + Y_1^4 + Y_2^4)$ are irreducible smooth curves in $\mathbb{P}^2$ provided $\mathrm{char}(k) \neq 2$, and that $\phi\colon (X_i) \mapsto (X_i^2)$ is a morphism from V to W. Determine the degree of ϕ, and compute e_P for all $P \in V$.

7. Show that the plane cubic $V = Z(F)$, $F = X_0X_2^2 - X_1^3 + 3X_1X_0^2$ is smooth if $\mathrm{char}(k) \neq 2, 3$. Find the degree and ramification degrees for (i) the projection $\phi = (X_0 : X_1)\colon V \to \mathbb{P}^1$ (ii) the projection $\phi = (X_0 : X_2)\colon V \to \mathbb{P}^1$.

8. Show that the Finiteness Theorem fails in general for a morphism of smooth affine curves.

   Let $V = Z(F) \subset \mathbb{P}^2$ be the curve given by $F = X_0X_2^2 - X_1^3$. Is V smooth? Show that $\phi\colon (Y_0 : Y_1) \mapsto (Y_0^3 : Y_0Y_1^2 : Y_1^3)$ defines a morphism $\mathbb{P}^1 \to V$ which is a bijection, but is not an isomorphism.

9. (i) Let $\phi = (1 : f)\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism given by a nonconstant polynomial $f \in k[t] \subset k(\mathbb{P}^1)$. Show that $\deg(\phi) = \deg f$, and determine the ramification points of ϕ — that is, the points $P \in \mathbb{P}^1$ for which $e_P > 1$. Do the same for a rational function $f \in k(t)$.

   (ii) Let $\phi = (t^2 - 7 : t^3 - 10)\colon \mathbb{P}^1 \to \mathbb{P}^1$. Compute $\deg(\phi)$ and e_P for all $P \in \mathbb{P}^1$.

   (iii) Let $f, g \in k[t]$ be coprime polynomials with $\deg(f) > \deg(g)$, and $\mathrm{char}(k) = 0$. Assume that every root of $f'g - g'f$ is a root of fg. Show that g is constant and f is a power of a linear polynomial.

   (iv) Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism in characteristic zero. Suppose that every ramification point $P \in \mathbb{P}^1$ satisfies $\phi(P) \in \{0, \infty\}$. Show that $\phi = (F_0^n : F_1^n)$ for some linear forms F_i. [Hint: choose coordinates so that $\phi(0) = 0$ and $\phi(\infty) = \infty$.]

   (v) Suppose $\mathrm{char}(k) = p \neq 0$, and let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be given by $t^p - t \in k(t)$. Show that ϕ has degree p and that it is only ramified at ∞.

10. Let $\phi\colon V \to W$ be a finite morphism of smooth projective irreducible curves, and $D = \sum n_Q Q$ a divisor on W. Define
$$\phi^* D = \sum_{P \in V} e_P n_{\phi(P)} P \in \mathrm{Div}(V).$$
Show that $\phi^*\colon \mathrm{Div}(W) \to \mathrm{Div}(V)$ is a homomorphism, that $\deg(\phi^* D) = \deg(\phi)\deg(D)$, and that if D is principal, so is $\phi^*(D)$. Thus ϕ^* induces a homomorphism $\mathrm{Cl}(W) \to \mathrm{Cl}(W)$.

11. (i) Use the Finiteness Theorem to show that if $\phi\colon V \to W$ is a morphism between smooth projective curves in characteristic zero which is a bijection, then ϕ is an isomorphism.

    (ii) Let k be algebraically closed of characteristic $p > 0$. Consider the morphism $\phi = (X_0^p : X_1^p)\colon V = \mathbb{P}^1 \to W = \mathbb{P}^1$. Show that ϕ is a bijection, $k(V)/\phi^* k(W)$ is purely inseparable of degree p, and that $e_P = p$ for every $P \in V$.

12. Let $V \subset \mathbb{P}^2$ be a plane curve defined by an irreducible homogeneous cubic. Show that if V is not smooth, then there exists a nonconstant morphism from $\mathbb{P}^1$ to V.

13. Let V be a smooth irreducible projective curve. Let $U \subset k(V)$ be a finite-dimension k-vector subspace of $k(V)$. Show that there exists a divisor D on V for which $U \subset \mathcal{L}(D)$.

14. Let V be a smooth irreducible projective curve, and $P \in V$ with $\ell(P) > 1$. Let $f \in \mathcal{L}(P)$ be nonconstant. Show that the rational map $(1 : f)\colon V \dashrightarrow \mathbb{P}^1$ is an isomorphism. Deduce that if V is a smooth projective irreducible curve which is not isomorphic to $\mathbb{P}^1$, then $\ell(D) \leq \deg D$ for any nonzero divisor D of positive degree.

15. Let V be a smooth plane cubic. Assume that V has equation $X_0 X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$, for some $\lambda \in k \setminus \{0, 1\}$.

    Let $P = (0 : 0 : 1)$ be the point at infinity in this equation. Writing $x = X_1/X_0$, $y = X_2/X_0$, show that x/y is a local parameter at P. [Hint: consider the affine piece $X_2 \neq 0$.] Hence compute $v_P(x)$ and $v_P(y)$. Show that for each $m \geq 1$, the space $\mathcal{L}(mP)$ has a basis consisting of functions x^i, $x^j y$, for suitable i and j, and that $\ell(mP) = m$.

16. Let $f \in k[x]$ a polynomial of degree $d > 1$ with distinct roots, and $V \subset \mathbb{P}^2$ the projective closure of the affine curve with equation $y^{d-1} = f(x)$. Assume that $\mathrm{char}(k)$ does not divide $d - 1$. Prove that V is smooth, and has a single point P at infinity. Calculate $v_P(x)$ and $v_P(y)$.

17. * Let $F(X_0, X_1, X_2)$ be an irreducible homogeneous polynomial of degree d, and let $X = Z(F) \subset \mathbb{P}^2$ be the curve it defines. Show that the degree of X is indeed d.

18. Let $\theta\colon V \to V$ be a surjective morphism from an irreducible projective variety V to itself, for which the induced map on function fields is the identity. Show that $\theta = id_V$.

    Now let V be a smooth irreducible projective curve and $\phi\colon V \to \mathbb{P}^1$ be a nonconstant morphism, such that $\phi^*\colon k(\mathbb{P}^1) \to k(V)$ is an isomorphism. Show that there exists a morphism $\psi\colon \mathbb{P}^1 \to V$ such that ψ^* is inverse to ϕ^*. Deduce that ϕ is an isomorphism.