Part II Algebraic Geometry

Example Sheet III, 2016

(For all questions, assume k is algebraically closed. Further, you can assume the characteristic is not equal to 2 if necessary. A * indicates a more difficult problem.)

1. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $x_1x_2^2 - x_3^3 \in k[x_0, \ldots, x_3]$. Find the dimension of the tangent space at all the singularities.
2. Consider $V = Z(I) \subset \mathbb{A}^3$ where I is generated by $x_1^3 - x_3$ and $x_2^2 - x_3$. Determine the points at which V is singular and compute the dimensions of the tangent spaces there.
3. Let $f, g : X \to Y$ be morphisms between algebraic varieties, and suppose there is a non-empty open subset $U \subseteq X$ such that $f|_U = g|_U$. Show $f = g$. [Hint: First reduce to the case $Y = \mathbb{P}^n$, and show that the map $f \times g : X \to \mathbb{P}^n \times \mathbb{P}^n$ is a morphism, where $f \times g(x_1, x_2) = (f(x_1), g(x_2))$. Next consider the diagonal $\Delta = \{(y, y) \mid y \in \mathbb{P}^n\} \subseteq \mathbb{P}^n \times \mathbb{P}^n$.]
4. Let X and Y be algebraic varieties. Consider the set of pairs (U, f) where $U \subseteq X$ is an open subset and $f : U \to Y$ is a morphism. Define a relation $(U, f) \sim (V, g)$ if $f|_{U \cap V} = g|_{U \cap V}$. Show this relation is an equivalence relation.

   We define a *rational map* $f : X \dashrightarrow Y$ between algebraic varieties to be an equivalence class of a pair (U, f).

   A rational map $f : X \dashrightarrow Y$ is *birational* if it admits a rational inverse, i.e., a rational map $g : Y \dashrightarrow X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$ as rational maps, where id_Y denotes the rational map $(Y, id_Y) : Y \dashrightarrow Y$.

   Show that the blow-up $\pi : X \to \mathbb{A}^n$ is a birational map.
5. Let $X = \{\varphi : k^2 \to k^3 \mid \varphi \text{ is linear, but } \textit{not} \text{ injective}\}$.
   (a) Show X is a Zariski closed subvariety of $\mathbb{A}^6$, hence an affine algebraic variety, and compute $A(X)$.
   (b) Find the singular points, if any, of X. Compute $d = \dim X$.
   (c) Show there is a birational map α from X to $\mathbb{A}^d$.
6. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2 x_2 = x_0^3$.
   (a) Show that the formula $(u, v) \mapsto (u^2 v, u^3, v^3)$ defines a morphism $\phi : \mathbb{P}^1 \to V$.
   (b) Write down a rational map $\psi : V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \setminus \{(0, 0, 1)\}$ which is inverse to ϕ on U. What is the geometric interpretation of ψ?
   (c) Show that ψ does not extend to a morphism at $(0, 0, 1)$.
7. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2 x_2 = x_0^2(x_0 + x_2)$. Find a surjective morphism $\phi : \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi(P) = \begin{cases} 2 & \text{if } P = (0, 0, 1) \\ 1 & \text{otherwise} \end{cases}$$

Is there a rational map $\psi\colon V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \setminus \{(0,0,1)\}$, which coincides with ϕ^{-1} on U?

8. Let V be the quadric $Z(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$, and H the plane $x_0 = 0$. Let $P = (1,0,0,0)$. Show that $\phi = (0, x_1, x_2, x_3)$ defines a rational map $\phi\colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H in $\phi(Q)$ whenever this is defined.

   Let $V_1 = V \cap \{x_1 = x_2\}$ and $L = H \cap \{x_1 = x_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\cong} L$.

9. Consider the birational map $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by (x_1x_2, x_0x_2, x_0x_1), and let $P_0 = (1,0,0)$, $P_1 = (0,1,0)$ and $P_2 = (0,0,1)$ be the points, at which ϕ is not a morphism. Let $L \subset \mathbb{P}^2$ be a line. Show that ϕ gives a morphism $L \to \mathbb{P}^2$ such that:
   (i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;
   (ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

   Determine the effect of ϕ on the cubic C with defining polynomial $x_0(x_1^2 + x_2^2) + x_1^2x_2 + x_1x_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

10. (a) Let $\phi : X \to Y$ be a morphism of affine varieties. Using the definition of tangent space in terms of the derivatives of elements of the ideal, show that for all $p \in X$, there is a linear map
$$d\phi : T_pX \to T_{\phi(p)}Y.$$
    (b) In the situation of (i), if ϕ is defined by an m-tuple of polynomials $(\Phi_1, \ldots, \Phi_m) \in A(X)^m$, write $d\phi$ in terms of the Φ_i.
    (c) Now assume that X and Y are arbitrary varieties. Using the definition of Zariski tangent space, show (i) in this more general context. Show the your answer coincides with your answer in (i).

11. * Show that if V is an irreducible plane curve with equation $x_0x_2^2 = x_1^3 + ax_0^2x_1 + bx_0^3$, then V is isomorphic to the variety $W \subset \mathbb{P}^3$ given by $x_0x_3 = x_1^2$, $x_2^2 = x_1x_3 + ax_0x_1 + bx_0^2$ via the map $\phi = (x_0^2, x_0x_1, x_0x_2, x_1^2)$.

12. Let $Y \subseteq \mathbb{A}^3$ be the surface given by the equation $x_1^2 + x_2^2 + x_3^2 = 0$. Consider the blow-up $X \subseteq \mathbb{A}^3 \times \mathbb{P}^2$ of $\mathbb{A}^3$, with $\varphi : X \to \mathbb{A}^3$ the projection and $E = \varphi^{-1}(0)$. Recall that the *proper transform* of Y is the closure of $\varphi^{-1}(Y) \setminus E$ in X. Describe the proper transform $\tilde{Y}$ of Y. Describe the fibres of the map $\varphi|_{\tilde{Y}} : \tilde{Y} \to Y$. Show that $\tilde{Y}$ is non-singular.