Part II Algebraic geometry

Example Sheet I, 2016

In all problems, you may assume that we are working over an algebraically closed field k.

1. Let $X \subseteq \mathbf{A}^n$ be an affine variety. Show, as discussed in lecture, that the two notions of regular function agree, i.e., $A(X) = \mathcal{O}_X(X)$. [Hint: You will need the Hilbert Nullstellensatz.]

2. Let $Y \subseteq \mathbf{A}^2$ be the curve given by $xy = 1$. Show that Y is not isomorphic to $\mathbf{A}^1$. Find all morphisms $\mathbf{A}^1 \to Y$ and $Y \to \mathbf{A}^1$.

3. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety, determine $I(Y)$, and show that $A(Y)$ is a polynomial ring in one variable. Y is called the *twisted cubic*.

4. Let $Y = Z(x^2 - yz, xz - x)$. Show that Y has 3 irreducible components. Describe them, and their corresponding prime ideals.

5. Show that any non-empty open subset of an irreducible algebraic set (i.e., a variety) is dense. Show that if an affine variety is Hausdorff, it consists of a single point.

Recall a *basis* for a topological space X is a collection $\mathcal{U}$ of open subsets of X such that (1) for every $x \subseteq X$, there is a $U \in \mathcal{U}$ with $x \in U$ and (2) for every $U_1, U_2 \in \mathcal{U}$ and $x \in U_1 \cap U_2$, there is a $U_3 \in \mathcal{U}$ such that $x \in U_3 \subseteq U_1 \cap U_2$. Show that if X is an affine variety, then the collection of open sets $\{X \setminus Z(f) \mid f \in A(X)\}$ forms a basis for the topology of X.

6. A topological space is called *Noetherian* if it satisfies the descending chain condition for closed subsets. Show that affine varieties are Noetherian in the Zariski topology.

7. Show that if $X \subseteq \mathbf{A}^n$, $Y \subseteq \mathbf{A}^m$ are affine varieties, then $X \times Y \subseteq \mathbf{A}^n \times \mathbf{A}^m = \mathbf{A}^{n+m}$ is a Zariski closed subset of $\mathbf{A}^{n+m}$. The more ambitious may try to show that $X \times Y$ is irreducible, but a similar result will be proved in lecture.

8. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t^3, t^4, t^5) \mid t \in k\}$. Show that Y is an affine variety, and determine $I(Y)$. Show $I(Y)$ cannot be generated by two elements.

9. Suppose the characteristic of k is not 2. Show that there are no non-constant morphisms from $\mathbf{A}^1$ to $E = Z(y^2 - x^3 + x) \subseteq \mathbf{A}^2$. [Hint: Consider the map $A(E) \to A(\mathbf{A}^1) = k[t]$, and the images of x and y under this map. Then use the fact that $k[t]$ is a UFD.]

10. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial, and consider $Y = Z(yf - 1) \subseteq \mathbf{A}^{n+1}$, with coordinates $x_1, \ldots, x_n, y$. Show that Y is irreducible. Show that the projection $\mathbf{A}^{n+1} \to \mathbf{A}^n$ given by $(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n)$ induces a morphism $Y \to \mathbf{A}^n$ which is a homeomorphism to its image $D(f) := \{(a_1, \ldots, a_n) \in \mathbf{A}^n \mid f(a_1, \ldots, a_n) \neq 0\}$. This gives the Zariski open set $D(f)$ the structure of an algebraic variety.

11. Show that $G = GL_n(k)$ is an affine variety, and that the multiplication and inverse maps are morphisms of algebraic varieties. We say G is an *affine algebraic group*.

12. Let $Mat_{n,m}$ be the set of n by m matrices with coefficients in k; this set can be identified with $\mathbf{A}^{nm}$ in the obvious way.

a) Show that the set of 2×3 matrices of rank ≤ 1 is an algebraic set.

b) Show that the matrices in $Mat_{n,m}$ of rank $\leq r$ is an algebraic set.

13. Let $f, g \in k[x, y]$ be polynomials, and suppose f and g have no common factor. Show there exists $u, v \in k[x, y]$ such that $uf + vg$ is a non-zero polynomial in $k[x]$.

Now let $f \in k[x, y]$ be irreducible. The variety $Z(f)$ is called an affine *plane curve*. Show that any proper subvariety of $Z(f)$ is finite.

14. Let A be a k-algebra. We say A is *finitely generated* if there is a surjective k-algebra homomorphism $k[x_1, \ldots, x_n] \to A$ for some n. Now suppose that A is a finitely generated k-algebra which is also an integral domain. Show that there is an affine variety Y with A isomorphic to $A(Y)$ as k-algebras.

15. Let $G = \mathbf{Z}/2\mathbf{Z}$ act on $k[x.y]$ by sending $x \mapsto -x$, $y \mapsto -y$. Show that the algebra of invariants $k[x, y]^G$ (the subring of polynomials left fixed by this action) defines an affine subvariety X of $\mathbf{A}^3$ by explicitly computing this ring of invariants. X is called the *rational double point*.

What is the relation of the points of X to the orbits of G acting on $\mathbf{A}^2$?