Part II Algebraic Geometry

Example Sheet III, 2015

(For all questions, assume k is algebraically closed. Further, you can assume the characteristic is not equal to 2 if necessary. A * indicates a more difficult problem.)

1. Determine the radical of the following ideals
   i) $(xy^3, x(x-y)) \subset k[x,y]$
   ii) $(xy^3, x^2(y-3)) \subset k[x,y]$
   iii) $(x^2(y-z), xy(y-z), xz(y-z)^2) \subset k[x,y,z]$
2. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $x_1x_2^2 - x_3^3 \in k[x_0, \ldots, x_3]$. Find the dimension of the tangent space at all the singularities.
3. Consider $V = Z(I) \subset \mathbb{A}^3$ where I is generated by $x_1^3 - x_3$ and $x_2^2 - x_3$. Determine the points at which V is singular and compute the dimensions of the tangent spaces there.
4. Let $X = \{\varphi : k^2 \to k^3 \mid \varphi \text{ is linear, but } \textit{not} \text{ injective}\}$.
   (a) Show X is a Zariski closed subvariety of $\mathbb{A}^6$, hence an affine algebraic variety, and compute $A(X)$.
   (b) Find the singular points, if any, of X. Compute $d = \dim X$.
   (c) Show there is a birational map α from X to $\mathbb{A}^d$.
5. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2x_2 = x_0^3$.
   (a) Show that the formula $(u,v) \mapsto (u^2v, u^3, v^3)$ defines a morphism $\phi : \mathbb{P}^1 \to V$.
   (b) Write down a rational map $\psi : V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \backslash \{(0,0,1)\}$ which is inverse to ϕ on U. What is the geometric interpretation of ψ?
   (c) Show that ψ does not extend to a morphism at $(0,0,1)$.
6. Let $V \subset \mathbb{P}^2$ be defined by $x_1^2x_2 = x_0^2(x_0 + x_2)$. Find a surjective morphism $\phi : \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi(P) = \begin{cases} 2 & \text{if } P = (0,0,1) \\ 1 & \text{otherwise} \end{cases}$$

   Is there a rational map $\psi \colon V \dashrightarrow \mathbb{P}^1$, a morphism on $U = V \setminus \{(0,0,1)\}$, which coincides with ϕ^{-1} on U?
7. Let V be the quadric $Z(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$, and H the plane $x_0 = 0$. Let $P = (1,0,0,0)$. Show that $\phi = (0, x_1, x_2, x_3)$ defines a rational map $\phi \colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H in $\phi(Q)$ whenever this is defined.
   *Show that ϕ is not a morphism.
   Let $V_1 = V \cap \{x_1 = x_2\}$ and $L = H \cap \{x_1 = x_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\cong} L$.
8. Consider the birational map $\phi \colon \mathbb{P}^2 \to \mathbb{P}^2$ given by (x_1x_2, x_0x_2, x_0x_1), and let $P_0 = (1,0,0)$, $P_1 = (0,1,0)$ and $P_2 = (0,0,1)$ be the points, at which ϕ is not a

morphism. Let $L \subset \mathbb{P}^2$ be a line. Show that ϕ gives a morphism $L \to \mathbb{P}^2$ such that:

(i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;

(ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

Determine the effect of ϕ on the cubic C with defining polynomial $x_0(x_1^2+x_2^2)+x_1^2x_2+x_1x_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

9. Let $\phi : X \to Y$ be a morphism of affine varieties.
   (a) Show that for all $p \in X$, there is a linear map
$$d\phi : T_pX \to T_{\phi(p)}Y.$$
   (b) If ϕ is defined by an m-tuple of polynomials $(\Phi_1, \ldots, \Phi_m) \in k[X]^m$, write $d\phi$ in terms of the Φ_i.
10. In this question, we will show that 'the generic hypersurface is smooth' — that is, that the set of smooth hypersurfaces of degree d is dense in the variety of all hypersurfaces of degree d in $\mathbb{A}^n$.

    Let $n, d \geq 1$, and let
$$X = \{f \in k[x_1, \ldots, x_n] \mid \deg f \leq d\},$$
and
$$Z = \left\{ (f,p) \in X \times \mathbb{A}^n \;\middle|\; \begin{array}{c} f(p) = 0 \text{ and } k[x_1,\ldots,x_n]/(f) \text{ is } \textit{not} \text{ the ring} \\ \text{of functions of a closed set which is smooth at } p \end{array} \right\}.$$
    (This is somewhat clumsy phrasing!)
   (a) Show $X \simeq \mathbb{A}^N$ for some N [you need not determine N] and that Z is a Zariski closed subvariety of $X \times \mathbb{A}^n$.
   (b) Show that the fibers of the projection map $Z \to \mathbb{A}^n$ are linear subspaces of dimension $N-(n+1)$.
   Conclude that $\dim Z = N - 1 < \dim X$.
   (c) Hence show that $\{f \in X \mid \deg f = d, Z(f) \text{ smooth }\}$ is dense in X.
   [Given your current state of knowledge about dimension, hand-waving about dimension calculations is acceptable at this point.]
11. * Show that if V is an irreducible plane curve with equation $x_0x_2^2 = x_1^3 + ax_0^2x_1 + bx_0^3$, then V is isomorphic to the variety $W \subset \mathbb{P}^3$ given by $x_0x_3 = x_1^2$, $x_2^2 = x_1x_3 + ax_0x_1 + bx_0^2$ via the map $\phi = (x_0^2, x_0x_1, x_0x_2, x_1^2)$.