An algebraic variety is a simplicial affine variety.

In this handout, we'll define an *arbitrary* algebraic variety as something built out of affine algebraic varieties by glueing. This is an easy special case of a general notion, and it's only this special case we'll explain. The definition of a variety is easy; the definition of a morphism is a little more subtle.

Varieties

First suppose given a topological space X, and open sets $U_1, \dots U_N$ of X such that the U_i cover X, i.e. $X = \cup_i U_i$.

Put $U_{ij} = U_i \cap U_j = U_{ji}$, and let $\phi_{ij} : U_{ij} \to U_i$ be the inclusion map. This is an open inclusion.

Write $U_. = \coprod U_i$, $U_{..} = \coprod U_{ij}$, and $U_{..} \rightrightarrows U_.$ as shorthand for this data. It may be helpful to notice $U_{..} = U_. \times_X U_.$.

We can recover X from this data, by glueing:

$$X = (\coprod U_i)/\sim,$$

where $\phi_{ij}(u) \sim \phi_{ji}(u)$, for all $u \in U_{ij}$.

This makes sense for any maps $U_{..} \rightrightarrows U_.$, and defines a sensible topological space X, as long as $\sim$ really is an equivalence relation, i.e. providing the maps

$$U_{ijk} := U_i \cap U_j \cap U_k \to U_i \cap U_j \to U_i, \qquad U_{ijk} := U_i \cap U_j \cap U_k \to U_i \cap U_k \to U_i$$

are the same.for all i, j, k.

Now, we can take this as the definition of an algebraic variety, by requiring that each of the spaces U_i, U_{ij}, U_{ijk} is a an affine variety, and that each of the maps $\phi_{ij} : U_{ij} \to U_i$, ϕ_{ijk} is an open inclusion (and in particular a morphism) of affine varieties,

We say that the data $\phi : U_{..} \rightrightarrows U_.$ defining X is a *presentation* of X.

Notice that nothing stops us from glueing the underlying sets U_i, with their Zariski toplogies, to get a topological space X. The additional 'algebraic' structure on X is encoded in the definition of a morphism.

Morphisms

Now, let $\phi : U_{..} \rightrightarrows U_.$ be a presentation of X, and $\psi : V_{..} \rightrightarrows V_.$ a presentation of Y, X and Y two algebraic varieties. To define a morphism of varieties is to define a morphism of presentations.

However, a map $f : X \to Y$ need not take one presentation to another. That is, it will often be the case that for some i, there is no j with $f(U_i) \subseteq V_j$. Nonetheless, sometimes we're lucky, and our map f really does preserve the cover.

Such maps are good, because we can insist $f|_{U_i} : U_i \to V_j$ is a morphism of *affine* varieties. Call such an f a "strict morphism".

Here is a definition of a strict morphism which doesn't mention the glued space X, and is written purely in terms of the presentation. (Rather than read it, it might be a better exercise to just invent it yourself).

A *strict morphism* $f : U_{\cdot} \to V_{\cdot}$ is a is a map $F : \{1, \dots, n\} \to \{1, \dots, m\}$, and morphisms $f_i : U_i \to V_{Fi}$, $f_{ij} : U_{ij} \to V_{Fi,Fj}$ of affine varieties such that for all i, j, $f_i \phi_{ij} = \psi_{Fi,Fj} f_{ij}$, and $f_{ij} = f_{ji}$.

So a strict morphism defines a map $f : X \to Y$ which preserves the presentations. It is clear how to compose strict morphisms. You should check that a strict morphism always defines a continuous map $f : X \to Y$.

A particular example of a strict morphism is a *refinement*, which is a strict morphism such that i) each f_i is a Zariski open embedding, ii) for all j, $\cup_{i:F(i)=j} f_i(U_i) = V_j$, and iii) $U_{ij} := U_i \cap U_j \simeq f_i U_i \cap f_j U_j \subseteq V_{Fi} \cap V_{Fj} = V_{FiFJ}$.

For example, if $X = V$ is affine, a refinement $f : U_{\cdot} \to X$ is just a presentation of V. In general a refinement of $V_{\cdot}$ is a presentation of each component V_j of $V_{\cdot}$, that is, just a presentation of the disconnected affine variety $V_{\cdot}$

For a concrete example, take $X = \mathbf{A}^1$, and $U_i = \mathbf{A}^1 \setminus \{p_i\}$, for distinct points $p_1, \dots, p_n$ of $\mathbf{A}^1$.

Now we want to define morphisms of algebraic varieties by considering refinements to be isomorphisms, and "formally adding their inverses" to the category. What that means here is:

Definition. *A morphism $F : U_{\cdot} \to V_{\cdot}$ is a pair of a refinement $\alpha : W_{\cdot} \to U_{\cdot}$, and strict morphism $f : W_{\cdot} \to V_{\cdot}$.*

If both α, f are refinements, then swapping the roles of α, f we get a morphism $G : V_{\cdot} \to U_{\cdot}$. Call such morphisms "simultaneous refinements".

Exercise. *Define composition of morphisms, and show that the isomorphisms are precisely the morphisms for which both α and f are refinements.*

Finally, we note that we're only interested in studying varieties upto isomorphism—that is we consider a refinement (of a cover of) the variety to be the same variety.

Definition. *Let X be an algebraic variety, given by a presentation $U_{\cdot\cdot} \rightrightarrows U_{\cdot}$. Say $U'_{\cdot}$ is an affine cover of X if there is a simultaneous refinement morphism $U'_{\cdot} \to U_{\cdot}$.*

So every affine cover of X defines a presentation of the variety X; all of these presentations are equally good.

Here is a concrete example. If $H_0, \dots, H_n$ are distinct hyperplanes with $\cap H_i = 0$ in a $n+1$-dimensional vector space V, then the sets $U_i = \mathbf{P}V \setminus \mathbf{P}H_i$ define a presentation of $\mathbf{P}V$. Show that if $H'_0, \dots H'_i$ are another such tuple of hyperplanes, then $U'_i = \mathbf{P}V \setminus \mathbf{P}H'_i$ defines an affine cover of $\mathbf{P}V$.

To see that we haven't done anything unexpected to morphisms of affine varieties, do the following easy

Exercise. *If $U_{\cdot}$ is a presentation of X, $V_{\cdot}$ a presentation of Y, and $F : X \to Y$ is a morphism of varieties in this sense, and X and Y are affine varieties, show F determines uniquely a morphism of affine varieties, and conversely.*

This data of a morphism isn't quite as horrible as it looks.

If you're give a map of sets $f : X \to Y$, f being continuous is a *property* of the map f.

Similarly, when given a map of sets $f : X \to Y$, where X and Y are algebraic varieties, then the property of f coming from a morphism is a property of the map of sets f, as the following attempts to show.

Exercise. *i) A morphism* $F : X \to Y$ *of varieties has the property that for every affine cover* $V_.$ *of* Y, $f^{-1}(V_j)$ *is an open subset of* X, *and an algebraic variety, and there exists an affine cover* $U(j)$ *of* $f^{-1}(V_j)$ *such that* $U_. = \coprod U(j)$ *is an affine cover of* X *and* $U_. \to V_.$ *is a strict morphism.*

ii) If X *and* Y *are algebraic varieties, a map of sets* $f : X \to Y$ *is a morphism if there is some affine cover* $V_.$ *of* Y *and affine cover* $U_.$ *of* X *which makes* f *a strict morphism.*

You should take part (ii) of the exercise as the practical definition of a morphism; the previous two pages were carefully checking it makes sense.