Algebraic Geometry, Part II, Example Sheet 2, 2014

Assume throughout that the base field k is algebraically closed.

1. Determine the radical of the following ideals

   i) $(xy^3, x(x-y))$

   ii) $(xy^3, x^2(y-3))$

   iii) $(x^2(y-z), xy(y-z), xz(y-z)^2)$

2. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $X_1X_2^2 - X_3^3 \in k[X_0, \ldots, X_3]$. Find the dimension of the tangent space at all the singularities.

3. Consider $V = Z(I) \subset \mathbb{A}^3$ where I is generated by $X_1^3 - X_3$ and $X_2^2 - X_3$. Determine the points at which V is singular and compute the dimensions of the tangent spaces there.

4. Show that the *affine quadric* $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum z_i^2 = 1\}$ is diffeomorphic to the tangent bundle of an $n-1$-sphere $TS^{n-1} = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid \sum x_i^2 = 1, \sum v_i x_i = 0\}$. [If you do not know what *diffeomorphic* means, just show they are homeomorphic.]

5. Let $X = \{\varphi : k^2 \to k^3 \mid \varphi$ is linear, but *not* injective$\}$.

   i) Show X is a Zariski closed subvariety of k^6, hence an affine algebraic variety, and compute $k[X]$.

   ii) Find the singular points, if any, of X. Compute $d = \dim X$.

   iv) Show there is a birational map α from X to k^d.

6. Let $V \subset \mathbb{P}^2$ be defined by $X_1^2 X_2 = X_0^3$.

   (a) Show that the formula $(u : v) \mapsto (u^2 v : u^3 : v^3)$ defines an morphism $\phi : \mathbb{P}^1 \to V$.

   (b) Write down a rational map $\psi \colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0:0:1)\}$ which coincides with ϕ^{-1} on U. What is the geometric interpretation of ψ?

   (c) Show that ψ is not regular at $(0:0:1)$.

7. Let $V \subset \mathbb{P}^2$ be defined by $X_1^2 X_2 = X_0^2(X_0 + X_2)$. Find a surjective morphism $\phi\colon \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi^{-1}(P) = \begin{cases} 2 & \text{if } P = (0:0:1) \\ 1 & \text{otherwise} \end{cases}$$

   Is there a rational map $\psi\colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0:0:1)\}$, which coincides with ϕ^{-1} on U?

8. Let V be the quadric $Z(X_0X_3 = X_1X_2) \subset \mathbb{P}^3$, and H the plane $X_0 = 0$. Let $P = (1:0:0:0)$. Show that $\phi = (0 : X_1 : X_2 : X_3)$ defines a rational map $\phi\colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H in $\phi(Q)$ whenever this is defined.

   *Show that ϕ is not a morphism.

   Let $V_1 = V \cap \{X_1 = X_2\}$ and $L = H \cap \{X_1 = X_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\sim} L$.

9. * (i) Repeat the previous question for $V = Z(I)$ where I is generated by

$$X_1^4 - X_2X_3, \quad X_1^3X_2 - X_3^2, \quad X_2^2 - X_1X_3$$

   * (ii) If you assumed $I = I(V)$ in (i), justify it.

10. Consider the birational map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ given by $(X_1X_2 : X_0X_2 : X_0X_1)$, and let $P_0 = (1 : 0 : 0)$, $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$ be the points, at which ϕ is not regular. Let $L \subset \mathbb{P}^2$ be a line. Show that ϕ gives a morphism $L \to \mathbb{P}^2$ such that:

    (i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;

    (ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

    Determine the effect of ϕ on the cubic C with defining polynomial $X_0(X_1^2 + X_2^2) + X_1^2X_2 + X_1X_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

11. Let $\phi : X \to Y$ be a morphism of affine varieties.

    (i) Show that for all $p \in X$, there is a linear map

$$d\phi : T_pX = Der(k[X], ev_p) \to T_{\phi(p)}Y = Der(k[Y], ev_{\phi(p)}).$$

    (ii) If ϕ is defined by an m-tuple of polynomials $(\Phi_1, \ldots, \Phi_m) \in k[X]^m$, write $d\phi$ in terms of the Φ_i.

    (iii) Deduce from (i) that if $\phi : X \to Y$ is a morphism of varieties, there is a linear map $d\phi : T_pX \to T_{\phi(p)}Y$.

12. In this question, we will show that 'the generic hypersurface is smooth' — that is, that the set of smooth hypersurfaces of degree d is dense in the variety of all hypersurfaces of degree d in $\mathbb{A}^n$

    Let $n, d \geq 1$, and let $X = \{f \in k[x_1, \ldots, x_n] \mid \deg f \leq d\}$, and $Z = \{(f, p) \in X \times \mathbb{A}^n \mid f(p) = 0$ and $k[x_1, \ldots, x_n]/(f)$ is *not* the ring of functions of an affine variety which is smooth at $p\}$.

    (This is somewhat clumsy phrasing!)

    i) Show $X \simeq \mathbb{A}^N$ for some N [you need not determine N] and that Z is a Zariski closed subvariety of $X \times \mathbb{A}^n$.

    ii) Show that the fibers of the projection map $Z \to \mathbb{A}^N$ are linear subspaces of dimension $N - (n + 1)$.

    Conclude that $\dim Z = N - 1 < \dim X$.

    iii) Hence show that $\{f \in X \mid \deg f = d, Z(f)$ smooth $\}$ is dense in X.

    [Quote any theorems of lectures you need].

13. * The *Krull dimension* of an irreducible variety X is the maximal length of a chain of irreducible Zariski closed subvarieties, ie the maximum n such that there are irreducible closed varieties $Z_0 \subset Z_1 \subset \cdots \subset Z_n$, $Z_i \neq Z_{i+1}$.

    Show that the Krull dimension of X is the transcendence dimension,

14. * Finish the proof of the following theorem about dimensions, which was stated in class.

    Let $\varphi : X \to Y$ be a morphism of affine algebraic varieties with X and Y irreducible, and suppose $\varphi(X)$ is dense in Y.

    Show that there is an affine open neighbourhood V of Y, such that $U = \varphi^{-1}(V)$ is affine open in X, and the map $\varphi : U \to V$ factors $U \to V \times \mathbb{A}^d \to V$ with the map $U \to V \times \mathbb{A}^d$ a *finite* map, and the map $V \times \mathbb{A}^d \to V$ is the natural projection.

    Conclude that for $v \in V$, the dimension of the fiber $\varphi^{-1}(v)$ is $\dim X - \dim Y$.

    Finally, observe that because this is true for X, Y affine, it is true for arbitrary irreducible quasi-projective varieties X and Y with $\varphi(X)$ dense in Y.