Part II Algebraic Geometry

Example Sheet I, 2012

1. i) Let Y be the curve $y = x^2$. Show $k[Y]$ is a polynomial algebra in one variable.

   ii) Let Y' be the curve $xy = 1$. Show $k[Y']$ is not isomorphic to $k[x]$, that is Y and Y' are not isomorphic. Find all elements of $Mor(Y, Y')$ and $Mor(Y', Y)$.

2. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show Y is an affine variety, determine $I(Y)$, and show $k[Y]$ is a polynomial algebra in one variable. Y is called the *twisted cubic*.

3. Let $Y = Z(x^2 - yz, xz - x)$. Show Y has 3 irreducible components. Describe them, and their prime ideals.

4. Show that if $X \subset \mathbf{A}^n$, and $Y \subset \mathbf{A}^m$ are Zariski closed subvarieties, then $X \times Y \subset \mathbf{A}^{n+m}$ is a Zariski closed subvariety, by explicitly writing $I(X \times Y)$ in terms of $I(X) = (f_1(x_1, \ldots, x_n), \ldots, f_t(x_1, \ldots, x_n))$ and $I(Y) = (h_1(y_1, \ldots, y_m), \ldots, h_s(y_1, \ldots, y_m))$.

   Show that the Zariski topology on $\mathbf{A}^2 = \mathbf{A}^1 \times \mathbf{A}^1$ is not the product topology of the Zariski topologies on $\mathbf{A}^1$.

5. Show that any non-empty open subset of an irreducible variety is dense. Show that if an affine variety is Hausdorff, it is a finite set of points.

6. A topological space is called *Noetherian* if it satisfies the descending chain condition for closed subsets. Show that affine algebraic varieties, with the Zariski topology, are Noetherian.

7. Let X be a topological space, and write $C(X)$ for the algebra of continuous functions from X to $\mathbf{C}$. Define maps Z, I between subsets of X and ideals of $C(X)$. Suppose X has the property that for every closed set F, and $p \notin F$, there exists a $f \in C(X)$ such that $f(F) = 0$ and $f(p) = 1$.

   Show that in this case $Z(I(F)) = F$ if F is closed, and so the map I defines an injection from closed subsets to ideals.

   Show i) any subset of $\mathbf{R}^n$, ii) any metric space, and iii) the Zariski topology on an affine algebraic variety (with $C(X)$ replaced by $k[X]$) all have this property.

   [Remark: There is an analogue of the Nullstellensatz, due to Gelfand-Naimark, which works for locally compact Hausdorff spaces.]

8. Let $Y \subseteq \mathbf{A}^3$ be the set $\{(t^3, t^4, t^5) \mid t \in k\}$. Show Y is an affine variety, and determine $I(Y)$. Show $I(Y)$ cannot be generated by two elements.

9. Show there are no non-constant morphisms from $\mathbf{A}^1$ to $E = Z(y^2 - x^3 + x)$.

10. Show that one can not make $\mathbf{A}^2 \backslash \{(0,0)\}$ into an affine variety in such a way that the inclusion map $\mathbf{A}^2 \backslash \{(0,0)\} \hookrightarrow \mathbf{A}^2$ is a morphism of affine varieties.

11. Show that $G = GL_n(k)$ is an affine variety, and that the multiplication and inverse maps are morphisms of algebraic varieties. We say G is an *affine algebraic group*. Show that if G is an affine algebraic group, and H is a subgroup which is also a closed subvariety of G, then H is also an affine algebraic group.

    Hence show $SL_n(k)$, $O_n(k) = \{A \mid AA^T = I\}$, and the group of invertible upper triangular matrices are also affine algebraic groups.

12. Let $Mat_{n,m}$ denote the set of n by m matrices with coefficients in k; this is an affine variety isomorphic to $\mathbf{A}^{nm}$.

    i) Show that the set of 2 by 3 matrices of rank ≤ 1 is an affine variety.

    ii) Show that the matrices of rank 2 in $Mat_{2,3}$ is a Zariski open subset, but not an affine variety. [Hint: You may do this directly, as in Q10, or you may deduce it from Q10, by finding a morphism $\mathbf{A}^2 \to Mat_{2,3}$ which takes the origin to a rank one matrix, and all other points to rank 2 matrices.]

    iii) Show that matrices in $Mat_{n,m}$ of rank $\leq r$ is an affine subvariety.

13. Show that the set of n by n matrices with distinct eigenvalues is an affine variety. Write its ring of functions explicitly when $n = 2$.

14. Let $f, g \in k[x, y]$ be polynomials, and suppose f and g have no common factor. Show there exists $u, v \in k[x, y]$ such that $uf + vg$ is a non-zero polynomial in $k[x]$.

    Now let $f \in k[x, y]$ be irreducible. The variety $Z(f)$ is called an affine *plane curve*. Show that any proper subvariety of $Z(f)$ is finite.

15. Let $G = \mathbf{Z}/2$ act on $k[x, y]$ by sending $x \mapsto -x$, $y \mapsto -y$. Show the algebra of invariants $k[x, y]^G$ defines an affine subvariety X of $\mathbf{A}^3$ by explicitly computing it in terms of generators and relations. X is called the *rational doublepoint.*

What is the relation of the points of X to the orbits of G on $\mathbf{A}^2$?

16*. You may assume $k = \mathbf{C}$ for this question.

Let Y be an affine variety, and G be a finite group. Suppose we are given an action on $k[Y]$ as algebra automorphisms. This implies each element of G acts on Y as a morphism. Show that the invariants of G, $k[Y]^G$ are the algebra of functions on an affine variety. Denote this variety Y/G, and show that the inclusion $k[Y]^G \hookrightarrow k[Y]$ gives a surjective morphism $Y \to Y/G$. Describe the fibers of this morphism.

1. Given distinct points $P_0, \cdots, P_{n+1}$ in $\mathbf{P}^n = \mathbf{P}(\mathbf{W})$, no $(n+1)$ of which are contained in a hyperplane, show that homogeneous coordinates may be chosen on $\mathbf{P}(\mathbf{W})$ so that $P_0 = (1\colon 0\colon \ldots\colon 0)$, $\cdots$, $P_n = (0\colon \ldots\colon 0\colon 1)$ and $P_{n+1} = (1\colon 1\colon \ldots\colon 1)$. [This generalises to arbitrary n a result you are very familiar with when $n = 1$.]

2. Given hyperplanes $H_0, \cdots, H_n$ of $\mathbf{P^n} = \mathbf{P}(\mathbf{W})$ such that $H_0 \cap \ldots \cap H_n = \emptyset$, show that homogeneous coordinates $x_0, \cdots, x_n$ can be chosen on $\mathbf{P}(\mathbf{W})$ such that each H_i is defined by $x_i = 0$.

3. Show that the set of hyperplanes in $\mathbf{P}(\mathbf{W})$ is parametrized by $\mathbf{P}(\mathbf{W}^*)$, where W^* is the dual vector space to W. If $P_1, \cdots, P_N$ are points of $\mathbf{P}(\mathbf{W})$, describe the set in $\mathbf{P}(\mathbf{W}^*)$ corresponding to hyperplanes not containing any of the P_i. Deduce (assuming k infinite) that there are infinitely many such hyperplanes.

4. Let V be a hypersurface in $\mathbf{P^n}$ defined by a non-constant homogeneous polynomial F, and L a (projective) line in $\mathbf{P^n}$; show that V and L must meet.

5. Prove that the decomposition of a variety into irreducible components is essentially unique. Decompose the projective variety V in $\mathbf{P^3}$ defined by equations $X_2^2 = X_1 X_3$, $X_0 X_3^2 = X_2^3$ into irreducible components.

6. Assume $\operatorname{char} k \neq 2$.

   i) Show that a homogeneous polynomial $F(X_0, X_1, X_2)$ of degree 2 can be written uniquely in the form $\mathbf{x}^T A \mathbf{x}$, where A is a 3×3 symmetric matrix with entries in k and $\mathbf{x}^T = (X_0, X_1, X_2)$; show that the polynomial is irreducible if and only if $\det(A) \neq 0$. Let $V \subset \mathbf{P}^2$ be the projective variety defined by the equation $F = 0$; if V is irreducible and k algebraically closed, show that you can choose coordinates such that $F = X_0^2 + X_1^2 + X_2^2$, and that V is isomorphic to $\mathbf{P}^1$.

   ii) In contrast, show that if $f(x, y) \in k[x, y]$ is an irreducible (non-homogeneous!) polynomial of degree 2, k algebraically closed, then $Z(f)$ is either $\mathbf{A}^1$ or k^*.

7. Consider the projective plane curves corresponding to the following affine curves in $\mathbf{A^2}$.

$(a)\ y = x^3$	$(b)\ xy = x^6 + y^6$
$(c)\ x^3 = y^2 + x^4 + y^4$	$(d)\ x^2 y + x y^2 = x^4 + y^4$
$(e)\ 2x^2 y^2 = y^2 + x^2$	$(f)\ y^2 = f(x)$ with f a polynomial of degree n.

   In each case, calculate the points at infinity of these curves, and find the singular points of the projective curve.

8. If $F(X_0, X_1, X_2)$ a homogeneous polynomial of degree $m > 0$, prove that $mF = \sum_{i=0}^{2} X_i\, \partial F / \partial X_i$. If F is irreducible and $V \subset \mathbf{P}^2$ is the projective plane curve defined by $F = 0$. Show that the singular locus of V consists precisely of the points P in $\mathbf{P^2}$ with $\partial F / \partial X_i(P) = 0$ for $i = 0, 1, 2$.