Algebraic Geometry, Part II, Example Sheet 2, 2011

Assume throughout that the base field k is algebraically closed.

1. Determine the singular points of the surface in $\mathbb{P}^3$ defined by the polynomial $X_1X_2^2 - X_3^3 \in k[X_0, \dots, X_3]$. Find the dimension of the tangent space at all the singularities.

2. Consider $V = Z(I) \subset \mathbb{A}^3$ where I is generated by $X_1^3 - X_3$ and $X_2^2 - X_3$. Determine the points at which V is singular and compute the dimensions of the tangent spaces there.

3. Show that the *affine quadric* $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum z_i^2 = 1\}$ is diffeomorphic to the tangent bundle of an $n-1$-sphere $TS^{n-1} = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid \sum x_i^2 = 1, \sum v_i x_i = 0\}$. [If you do not know what *diffeomorphic* means, just show they are homeomorphic.]

4. Let $X = \{\varphi : k^2 \to k^3 \mid \varphi \text{ is linear, but } not \text{ injective}\}$.

   i) Show X is a Zariski closed subvariety of k^6, hence an affine algebraic variety, and compute $k[X]$.

   ii) Find the singular points, if any, of X. Compute $d = \dim X$.

   iv) Show there is a birational map α from X to k^d.

5. Let $V \subset \mathbb{P}^2$ be defined by $X_1^2X_2 = X_0^3$.

   (a) Show that the formula $(u : v) \mapsto (u^2v : u^3 : v^3)$ defines an morphism $\phi : \mathbb{P}^1 \to V$.

   (b) Write down a rational map $\psi \colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0 : 0 : 1)\}$ which coincides with ϕ^{-1} on U. What is the geometric interpretation of ψ?

   (c) Show that ψ is not regular at $(0 : 0 : 1)$.

6. Let $V \subset \mathbb{P}^2$ be defined by $X_1^2X_2 = X_0^2(X_0 + X_2)$. Find a surjective morphism $\phi\colon \mathbb{P}^1 \to V$ such that, for $P \in V$,

$$\#\phi^{-1}(P) = \begin{cases} 2 & \text{if } P = (0 : 0 : 1) \\ 1 & \text{otherwise} \end{cases}$$

   Is there a rational map $\psi\colon V \dashrightarrow \mathbb{P}^1$, regular on $U = V \setminus \{(0 : 0 : 1)\}$, which coincides with ϕ^{-1} on U?

7. Let V be the quadric $Z(X_0X_3 = X_1X_2) \subset \mathbb{P}^3$, and H the plane $X_0 = 0$. Let $P = (1 : 0 : 0 : 0)$. Show that $\phi = (0 : X_1 : X_2 : X_3)$ defines a rational map $\phi\colon V \dashrightarrow H$ such that for $Q \in V$, the line PQ meets H in $\phi(Q)$ whenever this is defined.

   *Show that ϕ is not a morphism.

   Let $V_1 = V \cap \{X_1 = X_2\}$ and $L = H \cap \{X_1 = X_2\}$. Verify explicitly that ϕ induces an isomorphism $V_1 \xrightarrow{\sim} L$.

8. * (i) Repeat the previous question for $V = Z(I)$ where I is generated by

$$X_1^4 - X_2X_3, \quad X_1^3X_2 - X_3^2, \quad X_2^2 - X_1X_3$$

   * (ii) If you assumed $I = I(V)$ in (i), justify it.

9. Consider the birational map $\phi\colon \mathbb{P}^2 \to \mathbb{P}^2$ given by $(X_1X_2 : X_0X_2 : X_0X_1)$, and let $P_0 = (1 : 0 : 0)$, $P_1 = (0 : 1 : 0)$ and $P_2 = (0 : 0 : 1)$ be the points, at which ϕ is not regular. Let $L \subset \mathbb{P}^2$ be a line. Show that ϕ gives a morphism $L \to \mathbb{P}^2$ such that:

   (i) if $L \cap \{P_i\} = \emptyset$ then ϕ is an isomorphism of L with a conic in $\mathbb{P}^2$ which passes through all of the $\{P_i\}$;

   (ii) if L contains just one P_i then ϕ is an isomorphism of L with another line in $\mathbb{P}^2$

   Determine the effect of ϕ on the cubic C with defining polynomial $X_0(X_1^2 + X_2^2) + X_1^2X_2 + X_1X_2^2$. (Assume $\mathrm{char}(k) \neq 2$.) What happens to the singularity of C? Draw appropriate pictures.

10. Let $\phi : X \to Y$ be a morphism of affine varieties.

    (i) Show that for all $p \in X$, there is a linear map

    $$d\phi : T_pX = Der(k[X], ev_p) \to T_{\phi(p)}Y = Der(k[Y], ev_{\phi(p)}).$$

    (ii) If ϕ is defined by an m-tuple of polynomials $(\Phi_1, \dots, \Phi_m) \in k[X]^m$, write $d\phi$ in terms of the Φ_i.

    (iii) Deduce from (i) that if $\phi : X \to Y$ is a morphism of varieties, there is a linear map $d\phi : T_pX \to T_{\phi(p)}Y$.

11. * The *Krull dimension* of an irreducible variety X is the maximal length of a chain of irreducible Zariski closed subvarieties, ie the maximum n such that there are irreducible closed varieties $Z_0 \subset Z_1 \subset \cdots \subset Z_n$, $Z_i \neq Z_{i+1}$.

    Show that the Krull dimension of X is the transcendence dimension,