Algebraic Geometry, Part II, Example Sheet 3,2009

Assume throughout that the base field k is algebraically closed. If it helps, feel free to assume throughout that it has characteristic zero.

1. Let P be a smooth point of the irreducible curve V. Show that if $f, g \in k(V)$ then $v_P(f+g) \geq \min(v_P(f), v_P(g))$, with equality if $v_P(f) \neq v_P(g)$.

2. If P is a smooth point of an irreducible curve V and $\pi_P \in \mathcal{O}_{V,P}$ is a local parameter at P, show that $\dim_k \mathcal{O}_{V,P}/(\pi_P^n) = n$ for every $n \in \mathbb{N}$.

3. Show that $V = Z(X_0^8 + X_1^8 + X_2^8)$ and $W = Z(Y_0^4 + Y_1^4 + Y_2^4)$ are irreducible smooth curves in $\mathbb{P}^2$ provided $\mathrm{char}(k) \neq 2$, and that $\phi\colon (X_i) \mapsto (X_i^2)$ is a morphism from V to W. Determine the degree of ϕ, and compute e_P for all $P \in V$.

4. Show that the plane cubic $V = Z(F)$, $F = X_0X_2^2 - X_1^3 + 3X_1X_0^2$ is smooth if $\mathrm{char}(k) \neq 2, 3$. Find the degree and ramification degrees for (i) the projection $\phi = (X_0 : X_1)\colon V \to \mathbb{P}^1$ (ii) the projection $\phi = (X_0 : X_2)\colon V \to \mathbb{P}^1$.

5. Show that the Finiteness Theorem fails in general for a morphism of smooth affine curves.

   Let $V = Z(F) \subset \mathbb{P}^2$ be the curve given by $F = X_0X_2^2 - X_1^3$. Is V smooth? Show that $\phi\colon (Y_0 : Y_1) \mapsto (Y_0^3 : Y_0Y_1^2 : Y_1^3)$ defines a morphism $\mathbb{P}^1 \to V$ which is a bijection, but is not an isomorphism.

6. (i) Let $\phi = (1 : f)\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism given by a nonconstant polynomial $f \in k[t] \subset k(\mathbb{P}^1)$. Show that $\deg(\phi) = \deg f$, and determine the ramification points of ϕ — that is, the points $P \in \mathbb{P}^1$ for which $e_P > 1$. Do the same for a rational function $f \in k(t)$.

   (ii) Let $\phi = (t^2 - 7 : t^3 - 10)\colon \mathbb{P}^1 \to \mathbb{P}^1$. Compute $\deg(\phi)$ and e_P for all $P \in \mathbb{P}^1$.

   (iii) Let $f, g \in k[t]$ be coprime polynomials with $\deg(f) > \deg(g)$, and $\mathrm{char}(k) = 0$. Assume that every root of $f'g - g'f$ is a root of fg. Show that g is constant and f is a power of a linear polynomial.

   (iv) Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism in characteristic zero. Suppose that every ramification point $P \in \mathbb{P}^1$ satisfies $\phi(P) \in \{0, \infty\}$. Show that $\phi = (F_0^n : F_1^n)$ for some linear forms F_i. [Hint: choose coordinates so that $\phi(0) = 0$ and $\phi(\infty) = \infty$.]

   (v) Suppose $\mathrm{char}(k) = p \neq 0$, and let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be given by $t^p - t \in k(t)$. Show that ϕ has degree p and that it is only ramified at ∞.

7. Let $\phi\colon V \to W$ be a finite morphism of smooth projective irreducible curves, and $D = \sum n_Q Q$ a divisor on W. Define
$$\phi^* D = \sum_{P \in V} e_P n_{\phi(P)} P \in \mathrm{Div}(V).$$
   Show that $\phi^*\colon \mathrm{Div}(W) \to \mathrm{Div}(V)$ is a homomorphism, that $\deg(\phi^* D) = \deg(\phi)\deg(D)$, and that if D is principal, so is $\phi^*(D)$. Thus ϕ^* induces a homomorphism $\mathrm{Cl}(W) \to \mathrm{Cl}(W)$.

8. (i) Use the Finiteness Theorem to show that if $\phi\colon V \to W$ is a morphism between smooth projective curves in characteristic zero which is a bijection, then ϕ is an isomorphism.

   (ii) Let k be algebraically closed of characteristic $p > 0$. Consider the morphism $\phi = (X_0^p : X_1^p)\colon V = \mathbb{P}^1 \to W = \mathbb{P}^1$. Show that ϕ is a bijection, $k(V)/\phi^* k(W)$ is purely inseparable of degree p, and that $e_P = p$ for every $P \in V$.

9. Let $V \subset \mathbb{P}^2$ be a plane curve defined by an irreducible homogeneous cubic. Show that if V is not smooth, then there exists a nonconstant morphism from $\mathbb{P}^1$ to V.

10. Let V be a smooth irreducible projective curve. Let $U \subset k(V)$ be a finite-dimension k-vector subspace of $k(V)$. Show that there exists a divisor D on V for which $U \subset \mathcal{L}(D)$.

11. Let V be a smooth irreducible projective curve, and $P \in V$ with $\ell(P) > 1$. Let $f \in \mathcal{L}(P)$ be nonconstant. Show that the rational map $(1 : f)\colon V \dashrightarrow \mathbb{P}^1$ is an isomorphism. Deduce that if V is a smooth projective irreducible curve which is not isomorphic to $\mathbb{P}^1$, then $\ell(D) \leq \deg D$ for any nonzero divisor D of positive degree.

12. Let P be the point at infinity on $\mathbb{P}^1$ and $D = 4P$. Investigate the morphism ϕ_D. Show that there exists a smooth curve $V \subset \mathbb{P}^3$ of degree 4 which is isomorphic to $\mathbb{P}^1$.

13. Let V be a smooth plane cubic. Assume that V has equation $X_0X_2^2 = X_1(X_1 - X_0)(X_1 - \lambda X_0)$, for some $\lambda \in k \setminus \{0, 1\}$.

    Let $P = (0 : 0 : 1)$ be the point at infinity in this equation. Writing $x = X_1/X_0$, $y = X_2/X_0$, show that x/y is a local parameter at P. [Hint: consider the affine piece $X_2 \neq 0$.] Hence compute $v_P(x)$ and $v_P(y)$. Show that for each $m \geq 1$, the space $\mathcal{L}(mP)$ has a basis consisting of functions x^i, $x^j y$, for suitable i and j, and that $\ell(mP) = m$.

14. Let $f \in k[x]$ a polynomial of degree $d > 1$ with distinct roots, and $V \subset \mathbb{P}^2$ the projective closure of the affine curve with equation $y^{d-1} = f(x)$. Assume that $\mathrm{char}(k)$ does not divide $d - 1$. Prove that V is smooth, and has a single point P at infinity. Calculate $v_P(x)$ and $v_P(y)$.

    * Deduce (without using Riemann–Roch) that if $n > d(d-3)$, then $\ell((n+1)P) = \ell(nP) + 1$.

15. Let $F(X_0, X_1, X_2)$ be an irreducible homogeneous polynomial of degree d, and let $X = Z(F) \subset \mathbb{P}^2$ be the curve it defines. Show that the degree of X is indeed d.

16. A smooth irreducible projective curve V is covered by two affine pieces (with respect to different embeddings) which are affine plane curves with equations $y^2 = f(x)$ and $v^2 = g(u)$ respectively, with f a square-free polynomial of even degree $2n$ and $u = 1/x$, $v = y/x^n$ in $k(V)$. Determine the polynomial $g(u)$ and show that the canonical class on V has degree $2n - 4$. Why can we not just say that V is the projective plane curve associated to the affine curve $y^2 = f(x)$?

17. Let $V_0 \subset \mathbb{A}^2$ be the affine curve with equation $y^3 = x^4 + 1$, and let $V \subset \mathbb{P}^2$ be its projective closure. Show that V is smooth, and has a unique point Q at infinity. Let ω be the rational differential dx/y^2 on V. Show that $v_P(\omega) = 0$ for all $P \in V_0$. prove that $v_Q(\omega) = 4$ and hence that ω, $x\omega$ and $y\omega$ are all regular on V.

18. Let $\theta\colon V \to V$ be a surjective morphism from an irreducible projective variety V to itself, for which the induced map on function fields is the identity. Show that $\theta = id_V$.

    Now let V be a smooth irreducible projective curve and $\phi\colon V \to \mathbb{P}^1$ be a nonconstant morphism, such that $\phi^*\colon k(\mathbb{P}^1) \to k(V)$ is an isomorphism. Show that there exists a morphism $\psi\colon \mathbb{P}^1 \to V$ such that ψ^* is inverse to ϕ^*. Deduce that ϕ is an isomorphism.