**STATISTICS** Part IB
**Example Sheet 2 (of 3)** Lent 2026

1. Let $X$ have density function
$$f(x \mid \theta) = \frac{\theta}{(x+\theta)^2}, \quad x > 0,$$
where $\theta \in (0, \infty)$ is an unknown parameter. Find the likelihood ratio test of size 0.05 of $H_0 : \theta = 1$ against $H_1 : \theta = 2$ and show that the probability of Type II error is 19/21.

2. Let $X \sim N(\mu, 1)$ where $\mu$ is unknown. Find the most powerful tests of sizes 0.05 and 0.01 for the following hypotheses:
(a) $H_0 : \mu = 0$ vs $H_1 : \mu = 4$.
(b) $H_0 : \mu = 4$ vs $H_1 : \mu = 0$.
Explain how to interpret your results when the realised value is $X(\omega) = 2.1$.

3. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent, with $X_1, \ldots, X_n \sim \mathrm{Exp}(\theta_1)$ and $Y_1, \ldots, Y_n \sim \mathrm{Exp}(\theta_2)$. Recalling the forms of the relevant MLEs from Sheet 1, show that the likelihood ratio of $H_0 : \theta_1 = \theta_2$ and $H_1 : \theta_1 \neq \theta_2$ is a monotone function of $|T - 1/2|$, where
$$T = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i}.$$
By writing down the distribution of $T$ under $H_0$, express the likelihood ratio test of size $\alpha$ in terms of $|T - 1/2|$ and the quantiles of a beta distribution.

4. A machine produces biodegradable plastic articles (many of which are defective) in bunches of three articles at a time. Under the null hypothesis that each article has a constant (but unknown) probability $\theta$ of being defective, write down the probabilities $p_i(\theta)$ of a bunch having $i$ defective articles, for $i = 0, 1, 2, 3$. In a trial run in which 512 bunches were produced, the numbers of bunches with $i$ defective articles were 213 ($i = 0$), 228 ($i = 1$), 57 ($i = 2$) and 14 ($i = 3$). Carry out Pearson's chi-squared test at the 5% level of the null hypothesis, explaining carefully why the test statistic should be referred to the $\chi_2^2$ distribution.

5. Let $f_0$ and $f_1$ be probability mass functions on a countable set $\mathcal{X}$. State and prove a version of the Neyman–Pearson lemma for a size $\alpha$ test of $H_0 : f = f_0$ against $H_1 : f = f_1$ assuming that $\alpha$ is such that there exists a likelihood ratio test of exact size $\alpha$.

6. A random sample of 59 people from the planet Krypton yielded the results below.

| | | Eye-colour | |
|---|---|---|---|
| | | Blue | Brown |
| Sex | Male | 19 | 10 |
| | Female | 9 | 21 |

Carry out a Pearson's chi-squared test at the 5% level of the null hypothesis that sex and eye-colour are independent factors on Krypton. Now carry out the corresponding test at the 5% level of the null hypothesis that each of the cell probabilities is equal to 1/4. Comment on your results.

7. Write down from lectures the model and hypotheses for a test of homogeneity in a two-way contingency table. By first deriving the MLEs under each hypothesis, show that the likelihood ratio and Pearson's chi-squared tests are identical to those for the independence test. Apply the homogeneity test to the data below from a clinical trial for a drug, obtained by randomly allocating 150 patients to three equal groups (so the row totals are fixed).

| | Improved | No difference | Worse |
|---|---|---|---|
| Placebo | 18 | 17 | 15 |
| Half dose | 20 | 10 | 20 |
| Full dose | 25 | 13 | 12 |

8. Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\theta)$. Find the likelihood ratio test of size $\alpha$ of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$ and derive an expression for the power function. Is the test uniformly most powerful for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$? Is it uniformly most powerful for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$?

9. If $X \sim N(0,1)$ and $Y \sim \chi^2_n$ are independent, we say that $T = X/\sqrt{Y/n}$ has a $t$-distribution with $n$ degrees of freedom and write $T \sim t_n$. Show that the probability density function of $T$ is

$$f_T(t) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \frac{1}{(n\pi)^{1/2}} \frac{1}{(1+t^2/n)^{(n+1)/2}}, \quad t \in \mathbb{R}.$$

10. Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} N(\mu, \sigma^2)$, where $\sigma^2$ is unknown, and suppose we are interested in testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Letting $\overline{X} = n^{-1} \sum_{i=1}^n X_i$ and $S_{XX} = \sum_{i=1}^n (X_i - \overline{X})^2$, show that the likelihood ratio can be expressed as

$$L_X(H_0, H_1) = \left(1 + \frac{T^2}{n-1}\right)^{n/2},$$

where $T = \frac{n^{1/2}(\overline{X} - \mu_0)}{\{S_{XX}/(n-1)\}^{1/2}}$. Determine the distribution of $T$ under $H_0$, and hence determine the size $\alpha$ likelihood ratio test.

11. Statisticians A and B obtain independent samples $X_1, \dots, X_{10}$ and $Y_1, \dots, Y_{17}$ respectively, both from a $N(\mu, \sigma^2)$ distribution with both $\mu$ and $\sigma$ unknown. They estimate $(\mu, \sigma^2)$ by $(\overline{X}, S_{XX}/9)$ and $(\overline{Y}, S_{YY}/16)$ respectively, where for example, $\overline{X} = \frac{1}{10} \sum_{i=1}^{10} X_i$ and $S_{XX} = \sum_{i=1}^{10} (X_i - \overline{X})^2$. Given that the values $\overline{X} = 5.5$ and $\overline{Y} = 5.8$, which statistician's estimate of $\sigma^2$ is more probable to have exceeded the true value by more than 50%? Find this probability (approximately) in each case.

12. **(Harder)** In Question 5, does there exist a version of the Neyman–Pearson lemma when a likelihood ratio test of exact size $\alpha$ does not exist?