Example sheet

Example sheet 2

Problem 1. Consider the problem

$$P: \text{minimize } \; 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \; \text{ subject to } \quad \begin{array}{rcl} x_1 + \;\; x_2 + 2x_3 + \;\; x_4 + 3x_5 & \geq & 4 \\ 2x_1 - 2x_2 + 3x_3 + \;\; x_4 + \;\; x_5 & \geq & 3 \\ x_1,\, x_2,\, x_3,\, x_4,\, x_5 & \geq & 0. \end{array}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.

Problem 2. Use the simplex algorithm to solve the linear program in Question 13 on example sheet 1. Let

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Initialize the algorithm from the basic feasible solution $x = (0, 0, 4, 4, 1)$.

Problem 3. Consider the problem in Question 13 on example sheet 1 and add the constraint $x_1 + 3x_2 \leq 6$. Apply the simplex algorithm putting x_2 into the basis in the first step. Show that the solution at $x_1 = 0$, $x_2 = 2$ is degenerate. Explain with a diagram what happens.

Problem 4. Apply the simplex algorithm to

$$P: \text{minimize } \; -x_1 - 3x_2 \; \text{ subject to } \quad \begin{array}{rcl} x_1 - 2x_2 & \leq & 4 \\ -x_1 + \;\; x_2 & \leq & 3 \\ x_1,\, x_2 & \geq & 0. \end{array}$$

Explain what happens with the help of a diagram.

Problem 5. Show the following properties of two-player zero-sum games:

(a) If the payoff matrix A is symmetric; i.e., it satisfies $A = -A^T$, then the value of the game is 0.

(b) Suppose the payoff matrix A is an $n \times n$ matrix with each row and column summing to s. Show that the game has value s/n. [*Hint:* Guess the optimal strategy.]

Problem 6. Two players fight a paint-gun duel: they face each other $2n-1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends; otherwise, both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the ith pace forward ($i = 0, 1, \dots, n-1$) is $(i+1)/n$. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired. Show that, if $n = 4$, the strategy of firing in round 2 is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $\left(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}\right)$.]

Problem 7. By considering the symmetric payoff matrix

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}$$

show that optimal strategies for a two-person zero-sum game are not necessarily unique. Find all the optimal strategies.

Problem 8. Consider a two-player zero-sum game with payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

The optimization problem for player 1 is

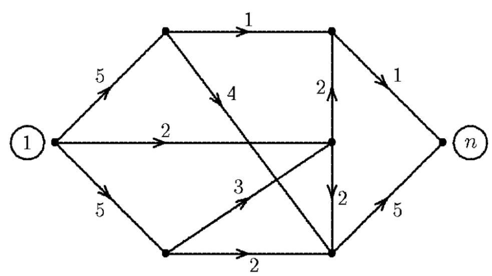
$$\begin{aligned} \text{maximize} \quad & v \\ \text{subject to} \quad & A^T p \geq ve, \\ & e^T p = 1 \\ & p \geq 0. \end{aligned}$$

(a) Setting $p = (p_1, 1-p_1)$, find the optimal strategies and the value of the game by drawing a picture as done in class.

(b) Argue that the value of the game $v > 0$, and by substituting $x = p/v$, show that optimization problem for the first player can be equivalently written as

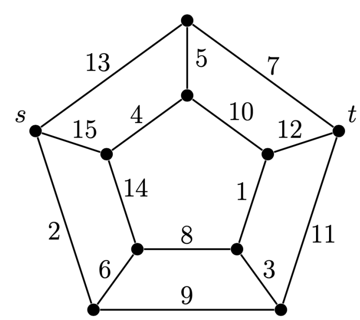
$$\begin{aligned} \text{minimize} \quad & e^T x \\ \text{subject to} \quad & A^T x \geq e \\ & x \geq 0. \end{aligned}$$

(c) Find the dual of the problem in part (b), solve it using the simplex method, and thereby identify the optimal strategies and the value of the game. Observe that solving the dual is more convenient since the basic feasible solution to initialize the simplex method is easily constructed.



Problem 9. Find a maximal flow and a minimal cut for the network pictured with a source at node 1 and a sink at node n.

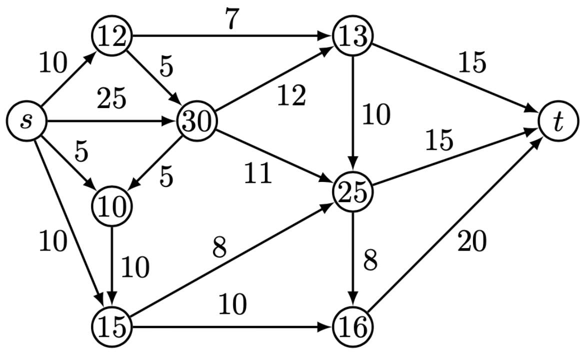
Problem 10. Explain how the Ford-Fulkerson algorithm can be used to find a maximum flow in an undirected network. Find a maximum flow from s to t in the following network, and prove that it is indeed optimal:



Problem 11. Consider a network with $2n+2$ nodes labelled $s, a_1, \ldots, a_n, b_1, \ldots, b_n, t$. Node s is the source, and node t is the sink. For each $i = 1, \ldots, n$, there is an edge (s, a_i) of capacity 1 from the source s to node a_i. For each $j = 1, \ldots, n$, there is an edge (b_j, t) of capacity 1 from node b_j to the sink t. All the other edges of the network are of the form (a_i, b_j) for some $i, j = 1, \ldots, n$ and have infinite capacity. Finally, suppose that for every subset $A \subseteq \{a_1, \ldots, a_n\}$ the number of nodes b_j such that there exists an edge (a_i, b_j) for some $a_i \in A$ is greater than or equal to $|A|$. Prove that the maximal flow has value n. (This is, essentially, Hall's marriage theorem.)

Problem 12. Suppose that a standard deck of 52 playing cards is dealt into 13 piles of 4 cards each. Show that it is possible to select exactly one card from each pile such that the selected cards contain one card of each of the 13 ranks Ace, 2, ..., 10, Jack, Queen, and King. (Consider a graph with 80 nodes, labeled $A, a_1, \ldots, a_{13}, b_1, \ldots, b_{52}, c_1, \ldots, c_{13}, B$. Let nodes $b_1, \ldots, b_{52}$ correspond to the 52 cards in the deck, $a_1, \ldots, a_{13}$ correspond to the 13 piles, and $c_1, \ldots, c_{13}$ correspond to the 13 card ranks. A node a_i is connected to a node b_j if pile i contains card b_j. A node b_j is connected to c_k if card b_j has rank k. A is connected to all a_i and B is connected to all c_i with capacity 1 edges. The capacities of all other edges is infinite. Show that max-flow and min-cut are equal to 13.)

Problem 13. Explain how a network flow problem can be augmented by constraints on the flow through a *vertex*. Now consider the network of one-way streets between locations s and t shown below, in which both streets and intersections are labeled with their capacities. Determine the



maximum flow from s to t. Suppose that the capacity constraint of one of the intersections could be removed completely by building a flyover. For which intersection should this be done in order to increase the maximum flow as much as possible?

Problem 14. Use the transportation algorithm to solve the problem given by the following tableau. Note that in finding an initial basic feasible solution, it may be beneficial to deviate from the procedure described in the lecture notes and instead look for a solution with small cost.

5	4	3	1	10
5	6	9	3	11
6	3	5	7	8
3	3	9	14	

Problem 15. A taxi company wants to send n taxis to pick up n customers, one per taxi, in a way that minimizes the sum of customers' waiting times. The time required by taxi i to pick up customer j is t_{ij}.

(a) Model this situation as an instance of the transportation problem. Which additional properties, if any, should a solution satisfy? Is the optimal solution guaranteed to satisfy these properties? Can the problem still be solved if the number of taxis exceeds the number of customers?

(b) What happens if we try to solve this problem with the transportation algorithm? Observe that a solution with overall waiting time zero is always optimal, and show that the set of optimal solutions does not change if we add or subtract the same value from all waiting times for a given taxi or customer. Use these insights to solve the problem for waiting times given by

$$T = \begin{pmatrix} 5 & 9 & 3 & 6 \\ 8 & 7 & 8 & 2 \\ 6 & 10 & 12 & 7 \\ 3 & 10 & 8 & 6 \end{pmatrix}.$$