Example sheet 1

Problem 1. Show that the set $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is convex.

Problem 2. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex. Show that the function $f : \mathcal{X} \to \mathbb{R}$ is convex if and only if the set $\{(x, y) \in \mathcal{X} \times \mathbb{R} : f(x) \leq y\}$ is convex.

Problem 3. Suppose the n functions $f_1, \ldots, f_n : \mathcal{X} \to \mathbb{R}$ are convex. Show that the function $g(x) = \max\{f_1(x), \ldots, f_n(x)\}$ is also convex.

Problem 4. Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_j(x) \leq b_j \text{ for } 1 \leq j \leq m \\ & x \in \mathcal{X}, \end{aligned}$$

with value function $\phi : \mathbb{R}^m \to \mathbb{R}$. Suppose that $\mathcal{X}$ is a convex set, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and the functions $h_j : \mathbb{R}^n \to \mathbb{R}$ are convex for each $1 \leq j \leq m$. Show that the value function ϕ is convex.

Problem 5. Suppose $f : \mathcal{X} \to \mathbb{R}$ is twice-differentiable and such that $\nabla^2 f(x)$ is positive definite for all $x \in \mathcal{X}$. Show that f is strictly convex. Find an example where $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex, but where $\nabla^2 f(x)$ is *not* positive definite for all $x \in \mathbb{R}^n$. [Consider the case $n = 1$.]

Problem 6. Let $f(x) = x^2 + \cos x$ for all $x \in \mathbb{R}$, and consider the problem of minimizing f.

(a) Show that the unique minimizer is $x^* = 0$.

(b) Find constants $0 < m < M$ and $L > 0$ such that $m \leq f''(x) \leq M$ and $|f'''(x)| \leq L$ for all x.

(c) Apply the gradient descent algorithm with step-size $t = 1/M$ and Newton's method as described in lectures. Starting with $x_0 = 1$, calculate x_1 and x_2 numerically in both cases. Tabulate $f(x_k) - f(x^*)$ for $k = 0, 1, 2$ for both methods. What do you observe?

(d) Repeat part (c) with $x_0 = 2\pi$. What do you observe now?

Problem 7. Given constants $p > 1$ and $a_1, \dots, a_n$ and $b \geq 0$, consider the problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{n} a_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} |x_i|^p \leq b. \end{aligned}$$

(a) Use the Lagrange multiplier method to solve the problem.

(b) Find the value function $\varphi(b)$ of this problem. Show that φ is convex.

(c) Find the dual problem and show that strong duality holds.

Problem 8. Suppose $0 < a_1 < a_2 < \cdots < a_n$. Show how to solve the problem

$$\text{mimimise} \sum_{i=1}^{n} \frac{1}{a_i + x_i} \text{ subject to } \sum_{i=1}^{n} x_i = b, x_i \geq 0 \text{ for } 1 \leq i \leq n.$$

Problem 9. Given constants $b_1,\ b_2$ such that $b_1 - e^{-b_2} \geq 0$ use the Lagrangian method to

$$\begin{aligned} \text{minimize} \quad & -2\tan^{-1} x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq b_1 \\ & -\log x_2 \leq b_2 \\ & x_1, x_2 \geq 0. \end{aligned}$$

[*Hint*: There will be two cases to check depending the constants b_1 and b_2.]

Problem 10. Given a $m \times n$ matrix A and a vector $b \in \mathbb{R}^n$, prove that x_0 is an extreme point of the set $\{x \in \mathbb{R}^n : \ Ax \leq b,\ x \geq 0\}$ if and only if $\binom{x_0}{z_0}$ is an extreme point of the set

$$\left\{ \binom{x}{z} \in \mathbb{R}^{n+m} : \ Ax + z = b,\ x \geq 0,\ z \geq 0 \right\}.$$

where $z_0 = b - Ax_0$.

Problem 11. Consider the following problems:

(a) minimize $c^T x$ subject to $Ax = b$

(b) minimize $c^T x$ subject to $Ax \leq b$

(c) minimize $c^T x$ subject to $Ax = b$, $x \geq 0$

(d) minimize $c^T x$ subject to $Ax \leq b$, $x \geq 0$

In each case

(i) Find the set Λ of values for the Lagrange multipliers λ for which the Lagrangian has a finite minimum.

(ii) For each value of $\lambda \in \Lambda$ calculate the minimum of the Lagrangian and write down the dual problem.

(iii) Write down the necessary and sufficient conditions for optimality.

(iv) Verify that the dual of the dual is the primal problem.

Problem 12. Suppose that a linear programming problem is written in the two equivalent forms

$$\text{minimize} \quad c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0,$$

where A is an $m \times n$ matrix, $c,\ x \in \mathbb{R}^n$; and

$$\text{minimize} \quad c_e^T x_e \quad \text{subject to} \quad A_e x_e = b,\ x_e \geq 0,$$

where, after the addition of slack variables and the extension of the matrix A and vector c in the appropriate way, A_e is $m \times (n+m)$, and $c_e,\ x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of the problem and show that the dual problems are equivalent to each other.

Problem 13. Consider the (primal) linear programming problem

$$P: \text{minimize } -x_1 - x_2 \text{ subject to } \begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 4 \\ x_1 - x_2 &\leq 1 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

(i) Solve P graphically in the x_1-x_2 plane.

(ii) Introduce slack variables x_3, x_4 x_5 and write the problem in standard form. How many basic solutions of the constraints are there? Determine the values of $x = (x_1, \ldots, x_5)^\top$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all the basic solutions non-degenerate?

(iii) Write down the dual problem in inequality form with variables λ_1, λ_2 and λ_3; add slack variables λ_4 and λ_5 and determine the values of $\lambda = (\lambda_1, \ldots, \lambda_5)^\top$ and of the dual objective function at each of the basic solutions to the dual. Which of these are feasible for the dual?

(iv) Show that for each basic solution x to the problem P there is exactly one basic solution λ to the dual giving the same values of the primal and dual objective functions and satisfying complementary slackness ($\lambda_i x_{i+2} = 0$, $i = 1, 2, 3$ and $x_j \lambda_{j+3} = 0$, $j = 1, 2$). For how many of these matched pairs (x, λ) is x feasible for the primal problem and λ feasible for the dual?

(v) On Example Sheet 2 you will be asked to solve this problem using the simplex algorithm. Initially, there are two possible choices of pivot column. Which will be the better choice if you wish to make the smallest number steps to reach the optimum?