Optimisation **Part IB - Easter 2018**

Example Sheet 2

Lecturer: Quentin Berthet

1. Consider the following linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 + 2x_2 \le 4\,, \\ & 2x_1 + x_2 \le 4\,, \\ & x_1 - x_2 \le 1\,, \\ & x_1, x_2 \ge 0\,. \end{aligned}$$

a) Solve the problem graphically in the plane.

b) Introduce slack variables x_3, x_4, and x_5 and write the problem in equality form. How many basic solutions are there? Determine the value of $x = (x_1, \dots, x_5)^\top$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all basic solutions non-degenerate?

c) Write down the dual problem in equality form using slack variables λ_4 and λ_5, and determine the value of $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ and of the objective function at each of the basic solutions of the dual. Which of these basic solutions are feasible?

d) Write down the complementary slackness conditions for the problem, and show that for each basic solution of the primal there is exactly one basic solution of the dual such that the two have the same value and satisfy complementary slackness. How many of these pairs are feasible for both primal and dual?

e) Solve the problem using the simplex method. Start from the basic feasible solution where $x_1 = x_2 = 0$, and try both choices for a variable to enter into the basis. How are the entries in the last row of the various tableaus related to the appropriate basic solutions of the dual?

2. Consider the three equations in six unknowns given by $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Remarks or errors can be addressed to q.berthet@statslab.cam.ac.uk

a) Let B $= \{1, 3, 6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$, where $x_B = (x_1, x_3, x_6)^\top$ and $x_N = (x_2, x_4, x_5)^\top$, and the matrices A_B and A_N consist of the appropriate columns of A. For $c \in \mathbf{R}^6$, write $c^\top x = c_B^\top x_B + c_N^\top x_N$ in terms of the non-basic variables by eliminating x_B.

b) Let $c = (3, 1, 3, 0, 0, 0)^\top$. Compute A_B^{-1} and the basic solution with basis B, and write $c^\top x$ in terms of the non-basic variables. Prove directly from the formula for $c^\top x$ that the basic solution with basis B maximizes $c^\top x$ subject to $Ax = b$ and $x \geq 0$.

3. Consider the linear program in question 1 on this sheet and augment it by the constraint $x_1 + 3x_2 \leq 6$. Use the simplex method to solve the new problem, letting x_2 enter the basis in the first round. Show that the basic solution where $x_1 = 0$ and $x_2 = 2$ is degenerate. Use a diagram to explain what happens.

4. Show that adding slack variables to a linear program does not change the extreme points of the feasible set of a problem, i.e. that $x^* \in \mathbf{R}^n$ is an extreme point of the set $\{x \in \mathbf{R}^n : x \geq 0, Ax \leq b\}$ if and only if for some $z^* \in \mathbf{R}^m$, $\binom{x^*}{z^*}$ is an extreme point of the set $\{\binom{x}{z} \in \mathbf{R}^{n+m} : \binom{x}{z} \geq 0, Ax + z = b\}$.

5. Give sufficient conditions for strategies x and y to be optimal for a matrix game with payoff matrix P and value v, and prove that they indeed guarantee optimality.

6. A matrix game with payoff matrix P is called symmetric if $P = -P^\top$. Show that symmetric matrix games have value 0, and that they may have multiple equilibria.

7. Consider two players who fight a duel. They start $2n + 1$ paces apart from each other, each with a single bullet. In each round, they simultaneously advance one pace and decide whether to fire a shot or not. The probability of hitting the opponent in round i is i/n. The game ends after n rounds or when a player is hit, whichever happens earlier. If exactly one player is hit, that player gets payoff -1, while the other player gets payoff 1. In all other cases both players get payoff 0. The guns are silent, so neither player knows before the end of the game whether the other has fired a shot. Show that firing a shot in round 2 is optimal if $n = 4$, while the mixed strategy $(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11})$ is optimal if $n = 5$.

8. Find all equilibria of the matrix game with payoff matrix

$$P = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}.$$

9. Consider a matrix game with payoff matrix $P \in \mathbf{R}^{n \times n}$, such that the entries in every row and every column of P sum to s. Show that the game has value s/n, for example by guessing a solution and showing that it is optimal.

10. Consider a matrix game with payoff matrix P.

a) Show that in finding the value and optimal strategies of the game, we may assume without loss of generality that $P > 0$ and thus $v > 0$. Simplify the LP for computing the value and optimal strategies by minimizing $1/v$ instead of maximizing v, and introducing new variables $z_i = x_i/v$ for $i = 1, \ldots, m$.

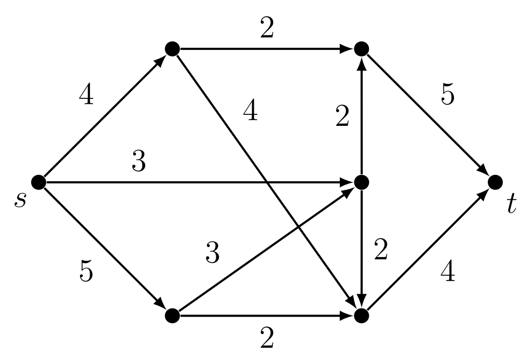
b) Propose an alternative method for finding optimal strategies that starts from given supports for these strategies, i.e. sets of actions played with positive probability, and uses the fact that all of them must yield the same expected payoff. Is this method restricted to matrix games, or can it be used to find equilibria in any game, assuming they exist?

c) Apply the two methods to the game with payoff matrix

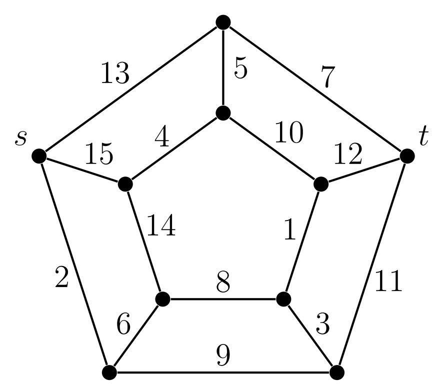
$$P = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix},$$

and discuss which of them requires a greater effort.

11. Find a maximum flow and a minimum cut of the following network with source s and sink t:



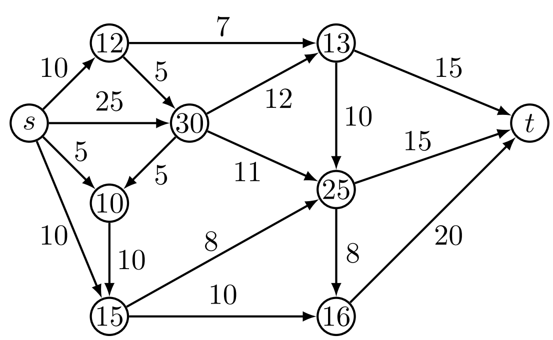
12. Explain how the Ford–Fulkerson algorithm can be used to find a maximum flow in an undirected network. Find a maximum flow from s to t in the following network, and prove that it is indeed optimal:



13. Show that a bipartite graph $G = (L \uplus R, E)$ with $|L| = |R|$ has a perfect matching if and only if $|N(X)| \geq |X|$ for every $X \subseteq L$, where $N(X) = \{j \in R : i \in X, (i,j) \in E\}$. (The implication in one direction is obvious. For the other direction, again consider the flow network discussed in the lecture notes, where a source s is connected to nodes in L and a sink t is connected to nodes in R, both via edges of capacity 1. Show that if G does not have a perfect matching, then this network has a cut S with $s \in S, t \in V \setminus S$, and $C(S) < |L|$. Let $L_S = L \cap S$, $R_S = R \cap S$, and $L_T = L \setminus S$, and show that the capacity of S is exactly $|L_T| + |R_S|$. Use this to prove that $|N(L_S)| < L_S$.)

14. Suppose that a standard deck of 52 playing cards is dealt into 13 piles of 4 cards each. Show that it is possible to select exactly one card from each pile such that the selected cards contain one card of each of the 13 ranks Ace, 2, ..., 10, Jack, Queen, and King. (Consider a graph with 80 nodes, labeled $A, a_1, \ldots, a_{13}, b_1, \ldots, b_{52}, c_1, \ldots, c_{13}, B$. For $i = 1, \ldots, 13$, add arcs (A, a_i) and (c_i, B), each of capacity 1. For each $i = 1, \ldots, 13$ add uncapacitated arcs (a_i, b_j) and (b_j, c_i) for $j \in B_i$, where $B_i \subseteq \{b_1, \ldots, b_{52}\}$ and $|B_i| = 4$, such that for all $j = 1, \ldots, n$, b_j has degree 2. Show that the size of a minimum cut separating A from B is 13.)

15. Explain how a network flow problem can be augmented by constraints on the flow through a *vertex*. Now consider the following network of one-way streets between locations s and t, in which both streets and intersections are labeled with their capacities:



Determine the maximum flow from s to t. Suppose that the capacity constraint of one of the intersections could be removed completely by building a flyover. For which intersection should this be done in order to increase the maximum flow as much as possible?

16. Use the transportation algorithm to solve the problem given by the following tableau:

5	4	3	1	10
5	6	9	3	11
6	3	5	7	8
3	3	9	14	

Note that in finding an initial basic feasible solution, it may be beneficial to deviate from the procedure described in the lecture notes and instead look for a solution with small cost.

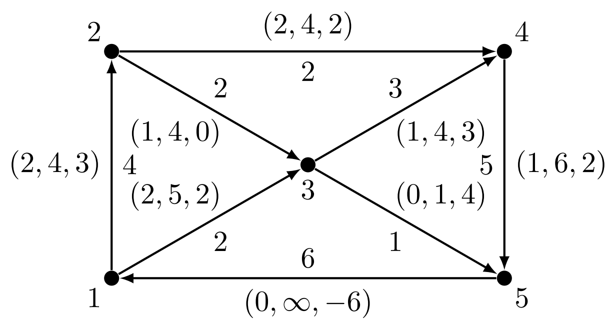
17. A taxi company wants to send n taxis to pick up n customers, one per taxi, in a way that minimizes the sum of customers' waiting times. The time required by taxi i to pick up customer j is t_{ij}.

a) Model this situation as an instance of the transportation problem (no pun intended). Which additional properties, if any, should a solution satisfy? Is the optimal solution guaranteed to satisfy these properties? Can the problem still be solved if the number of taxis exceeds the number of customers?

b) What happens if we try to solve this problem with the transportation algorithm? Observe that a solution with overall waiting time zero is always optimal, and show that the set of optimal solutions does not change if we add or subtract the same value from all waiting times for a given taxi or customer. Use these insights to solve the problem for waiting times given by

$$T = \begin{pmatrix} 5 & 9 & 3 & 6 \\ 8 & 7 & 8 & 2 \\ 6 & 10 & 12 & 7 \\ 3 & 10 & 8 & 6 \end{pmatrix}.$$

18. The transportation algorithm can be generalized to the so-called network simplex method, which is able to solve minimum-cost flow problems on arbitrary networks. Consider the following network (V, E), in which each edge is labeled with a flow x_{ij}, and a triple $(\underline{m}_{ij}, \overline{m}_{ij}, c_{ij})$ indicating the lower and upper bound and the cost for that edge:



Note that the current flow x is associated with a spanning tree (V, T) of the network such that $x_{ij} \in \{\underline{m}_{ij}, \overline{m}_{ij}\}$ for all $(i, j) \notin T$, and use this fact to find values for the dual variables λ_i. Find all edges that violate dual feasibility, and show that there are two ways of pushing flow along a cycle of the network such that overall cost is reduced. Now show that for one of them, pushing the maximum amount of flow results in an optimal solution.