Optimisation Michael Tehranchi

Example sheet 2 - Easter 2017

1. Consider the three equations in six unknowns given by $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}.$$

Choose $B = \{1,3,6\}$ and write $Ax = b$ in the form $A_B x_B + A_N x_N = b$ where $x_B = (x_1, x_3, x_6)^\top$, $x_N = (x_2, x_4, x_5)^\top$ and the matrices A_B and A_N are constructed appropriately.

Now write $c^\top x = c_B^\top x_B + c_N^\top x_N$ and hence write $c^\top x$ in terms of the matrices A_B, A_N and the variables x_N (i.e., eliminate x_B).

Compute A_B^{-1} and hence calculate the basic solution having B as basis. For $c = (3,1,3,0,0,0)^\top$ write $c^\top x$ in terms of the non-basic variables. Prove directly from the formula for $c^\top x$ that the basic solution that you have computed is optimal for the problem maximize $c^\top x$ subject to $Ax = b$, $x \geq 0$.

Compare your answer to your answer to Question 8 on example sheet 1 and confirm that the final tableau had rows corresponding to the equation $x_B + A_B^{-1} A_N x_N = A_B^{-1} b$.

2. Consider the problem in Question 7 on example sheet 1 and add the constraint $x_1 + 3x_2 \leq 6$. Apply the simplex algorithm putting x_2 into the basis at the first stage. Show that the solution at $x_1 = 0$, $x_2 = 2$ is degenerate. Try each of the possibilities for the variable leaving the basis. Explain, with a diagram, what happens.

3. Show that introducing slack variables in a linear programming problem does not change the extreme points of the feasible set by proving that x is an extreme point of the set $\{x : \ Ax \leq b, \ x \geq 0\}$ if and only if $\binom{x}{z}$ is an extreme point of the set

$$\left\{ \begin{pmatrix} x \\ z \end{pmatrix} : \ Ax + z = b, \ x \geq 0, \ z \geq 0 \right\}.$$

4. Give sufficient conditions for strategies p and q to be optimal for a two-person zero-sum game with payoff matrix A and value v.

Two players fight a duel: they face each other $2n - 1$ paces apart and each has a single bullet in his gun. At a signal each may fire. If either is hit or if both fire the game ends; otherwise, both advance one pace and may again fire. The probability of either hitting his opponent if he fires after the ith pace forward ($i = 0, 1, \ldots, n-1$) is $(i+1)/n$. If a player survives after his opponent has been hit his payoff is $+1$ and his opponent's payoff is -1. The payoff is 0 if neither or both are hit. The guns are silent so that neither knows whether or not his opponent has fired. Show that, if $n = 4$, the strategy 'shoot after taking one step' is optimal for both, but that if $n = 5$ a mixed strategy is optimal. [*Hint*: $\left(0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11}\right)$.]

5. By considering the payoff matrix

$$A = \begin{pmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \\ 0 & 3 & -4 & 0 \end{pmatrix}$$

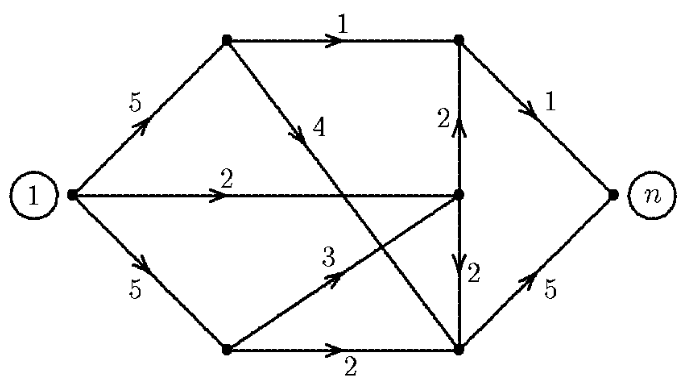
show that optimal strategies for a two-person zero-sum game are not necessarily unique. Find all the optimal strategies.

6. The $n \times n$ matrix of a two-person zero-sum game is such that the row and column sums all equal s. Show that the game has value s/n. [*Hint*: Guess a solution and show that it is optimal.]

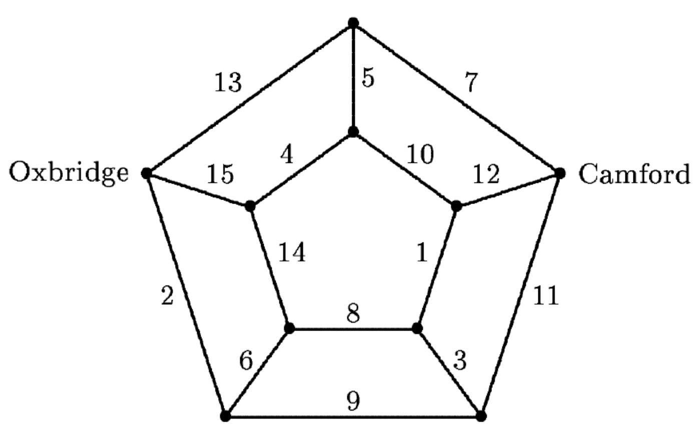
7. Find optimal strategies for both players and the value of the game which has payoff matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}.$$

8. Find a maximal flow and a minimal cut for the network pictured with a source at node 1 and a sink at node n.



9. Devise rules for a version of the Ford–Fulkerson algorithm which works with undirected arcs.

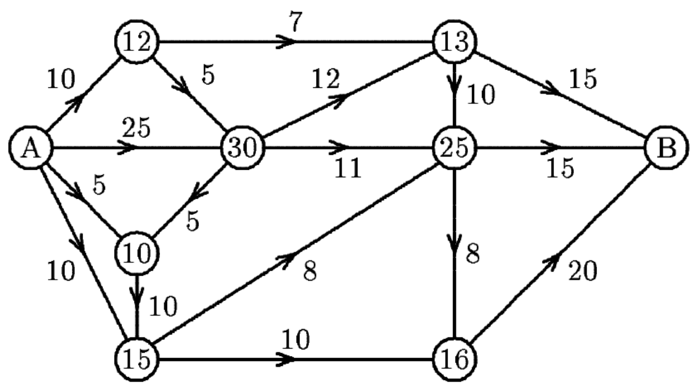


As a consequence of a drought, an emergency water supply must be pumped from Oxbridge to Camford along the network of pipes pictured below. The numbers against the pipes show their maximal capacities, and each pipe may be used in either direction. Find the maximal flow and prove that it is maximal.

10. How would you augment a directed network to incorporate restrictions on node capacity (the total flow permitted through a node) in maximal-flow problems?

The road network between two towns A and B pictured below. Each road is marked with an arrow giving the direction of the flow, and a number which represents its capacity. Each of the nodes of the graph represents a village. The total flow into a village cannot exceed its capacity (the number in the circle at the node). Obtain the maximal flow from A to B.

The Minister of Transport intends to build a by-pass around one of the villages, whose effect would be to completely remove the capacity constraint for that village. Which village should receive the by-pass if the intention is to increase the maximal flow from A to B by as much as possible? What would the new maximal flow be?



11. Consider a network with $2n+2$ nodes labelled $s, a_1, \dots, a_n, b_1, \dots, b_n, t$. Node s is the source, and node t is the sink. For each $i = 1, \dots, n$, there is an edge (s, a_i) of capacity 1 from the source s to node a_i. For each $j = 1, \dots, n$, there is an edge (b_j, t) of capacity 1 from node b_j to the sink t. All the other edges of the network are of the form (a_i, b_j) for some $i, j = 1, \dots, n$ and have infinite capacity. Finally, suppose that for ever subset $A \subseteq \{a_1, \dots, a_n\}$ the number of nodes b_j such that there exists an edge (a_i, b_j) for some $a_i \in A$ is greater than or equal to $|A|$. Prove that the maximal flow has value n. (This is, essentially, Hall's marriage theorem.)

12. Sources 1, 2, 3 stock candy floss in amounts of 20, 42, 19 tons respectively. The demand for candy floss at destinations 1, 2, 3 are 39, 34, 7 tons respectively. The matrix of transport costs per ton is

$$\begin{pmatrix} 7 & 4 & 9 \\ 8 & 12 & 5 \\ 3 & 11 & 7 \end{pmatrix}$$

with the (i, j) entry corresponding to the route $i \to j$. Find the optimal transportation scheme and the minimal cost by applying the transportation algorithm starting from (a) an assignment given by the NW method, and (b) an assignment given by the greedy algorithm.