

Optimisation

Michael Tehranchi

Example sheet 1 - Easter 2017

1. Given constants $p > 1$ and a_1, \dots, a_n , use the Lagrangian method to

$$\text{maximise } \sum_{i=1}^n a_i x_i \text{ subject to } \sum_{i=1}^n |x_i|^p \leq 1.$$

2. Given constants b_1, b_2 such that $b_1 - e^{-b_2} \geq 0$ use the Lagrangian method to

$$\text{maximise } 2 \tan^{-1} x_1 + x_2 \text{ subject to } x_1 + x_2 \leq b_1, -\ln x_2 \leq b_2, x_1 \geq 0, x_2 \geq 0.$$

[Hint: There will be two cases to check depending the constants b_1 and b_2 .]

3. Given non-negative constants p_1, \dots, p_n and q_1, \dots, q_n and $0 < \alpha < 1$, consider the problem to

$$\text{maximise } \sum_{i=1}^n p_i x_i \text{ subject to } \sum_{i=1}^n q_i x_i \leq \alpha, 0 \leq x_i \leq 1 \text{ for all } i.$$

Show that the feasible vector $x^* \in \mathbb{R}^n$ is optimal if there exists a $\lambda^* \geq 0$ such that

- $\sum_{i=1}^n q_i x_i^* = \alpha$,
- if $p_i > q_i \lambda^*$ then $x_i^* = 1$,
- if $p_i < q_i \lambda^*$ then $x_i^* = 0$.

4. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Show that for all real b the set $\{x : g(x) \leq b\}$ is convex. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and for every b let

$$\phi(b) = \inf\{f(x) : g(x) \leq b\}.$$

Assuming $\phi(b)$ is finite for all b , show that the function ϕ is convex.

5. Consider the following problems:

- (a) minimise $c^\top x$ subject to $Ax = b$;
- (b) minimise $c^\top x$ subject to $Ax \leq b$;
- (c) minimise $c^\top x$ subject to $Ax = b, x \geq 0$;
- (d) minimise $c^\top x$ subject to $Ax \leq b, x \geq 0$.

In each case

- (i) find the set Λ of values for the Lagrange multipliers λ for which the Lagrangian has a finite minimum (subject to the appropriate regional constraint, if any);
- (ii) for each value of $\lambda \in \Lambda$ calculate the minimum of the Lagrangian and write down the dual problem;
- (iii) write down the necessary and sufficient conditions for optimality;
- (iv) verify that the dual of the dual is the primal problem.

6. Suppose that a linear programming problem is written in the two equivalent forms

$$\text{minimise } c^\top x \text{ subject to } Ax \leq b, x \geq 0,$$

where A is an $m \times n$ matrix, $c, x \in \mathbb{R}^n$; and

$$\text{minimise } c_e^\top x_e \text{ subject to } A_e x_e = b, x_e \geq 0,$$

where, after the addition of slack variables and the extension of the matrix A and vector c in the appropriate way, A_e is $m \times (n + m)$, and $c_e, x_e \in \mathbb{R}^{n+m}$. Use your answers to the previous question to write down the dual problem to both versions of the problem and show that the dual problems are equivalent to each other.

7. Consider the (primal) linear programming problem

$$P : \text{maximise } x_1 + x_2 \text{ subject to } \begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 4 \\ x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(i) Solve P graphically in the x_1 - x_2 plane.

(ii) Introduce slack variables x_3, x_4, x_5 and write the problem in equality form. How many basic solutions of the constraints are there? Determine the values of $x = (x_1, \dots, x_5)^\top$ and of the objective function at each of the basic solutions. Which of the basic solutions are feasible? Are all the basic solutions non-degenerate?

(iii) Write down the dual problem in inequality form with variables λ_1, λ_2 and λ_3 ; add slack variables λ_4 and λ_5 and determine the values of $\lambda = (\lambda_1, \dots, \lambda_5)^\top$ and of the dual objective function at each of the basic solutions to the dual. Which of these are feasible for the dual?

(iv) Show that for each basic solution x to the problem P there is exactly one basic solution λ to the dual giving the same values of the primal and dual objective functions and satisfying complementary slackness ($\lambda_i x_{i+2} = 0, i = 1, 2, 3$ and $x_j \lambda_{j+3} = 0, j = 1, 2$). For how many of these matched pairs (x, λ) is x feasible for the primal problem and λ feasible for the dual?

(v) Solve the problem P using the simplex algorithm starting with the initial basic feasible solution $x_1 = x_2 = 0$. Try both choices of the variable to introduce into the basis on the first step. Compare the objective rows of the various tableaux generated with appropriate basic solutions of the dual problem. What do you observe?

8. Use the simplex algorithm to solve

$$P : \text{maximise } 3x_1 + x_2 + 3x_3 \text{ subject to } \begin{aligned} 2x_1 + x_2 + x_3 &\leq 2 \\ x_1 + 2x_2 + 3x_3 &\leq 5 \\ 2x_1 + 2x_2 + x_3 &\leq 6 \\ x_1, x_2, x_3 &\geq 0, \end{aligned}$$

Each row of the final tableau is the sum of scalar multiples of the rows of the initial tableau. Explain how to determine the scalar multipliers directly from the final tableau.

Let $P(\epsilon)$ be the linear programming problem when the right-hand side $b = (2, 5, 6)^\top$ is replaced by the perturbed vector $b(\epsilon) = (2 + \epsilon_1, 5 + \epsilon_2, 6 + \epsilon_3)^\top$. Give a formula, in terms of $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^\top$, for the optimal value for $P(\epsilon)$ when the ϵ_i are small. For what ranges of values for $\epsilon_1, \epsilon_2, \epsilon_3$ does your formula hold?

9. Apply the simplex algorithm to

$$P : \text{maximise } x_1 + 3x_2 \text{ subject to } \begin{aligned} x_1 - 2x_2 &\leq 4 \\ -x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Explain what happens with the help of a diagram.

10. Use the two-phase algorithm to solve:

$$\begin{array}{ll} \text{maximise} & -2x_1 - 2x_2 \text{ subject to} \\ & 2x_1 - 2x_2 \leq 1 \\ & 5x_1 + 3x_2 \geq 3 \\ & x_1, x_2 \geq 0. \end{array}$$

[*Hint:* You should get $x_1 = \frac{9}{16}$, $x_2 = \frac{1}{16}$. Note that it is possible to choose the first pivot column so that Phase I lasts only one step, but this requires a different choice of pivot column than the one specified by the usual rule-of-thumb.]

11. Use the two-phase algorithm to solve:

$$\begin{array}{ll} \text{minimise} & 13x_1 + 5x_2 - 12x_3 \text{ subject to} \\ & 2x_1 + x_2 + 2x_3 \leq 5 \\ & 3x_1 + 3x_2 + x_3 \geq 7 \\ & x_1 + 5x_2 + 4x_3 = 10 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

12. Consider the problem

$$\begin{array}{ll} \text{minimise} & 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \text{ subject to} \\ & x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \\ & 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Write down the dual problem, and solve this graphically. Hence deduce the optimal solution to the primal problem.