METRIC AND TOPOLOGICAL SPACES, SHEET II: 2019

1. Let $A \subset \mathbb{R}^n$ be a non-compact subset. Show that there exists a continuous function on A which is not bounded.
2. Show that the product of two connected spaces is connected.
3. Show that a continuous function $f : [0,1] \to [0,1]$ has a fixed point.
4. Let X be a topological space and $A \subset X$ a connected subspace of X. If K is a subspace containing A and contained in the closure of A, i.e. $A \subset K \subset \text{cl}(A)$, prove that K is connected.
5. Let X and Y be topological spaces and $f : X \to Y$ be a continuous bijection. Prove that if X is compact and Y is Hausdorff, then f is a homeomorphism. Give an examples to show that the Hausdorff condition is necessary.
6. Prove that there is no continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$.
7. Let X be a compact Hausdorff spaces with disjoint closed subspaces C_1 and C_2. Show there exist disjoint open sets U_1 and U_2 such that $C_i \subset U_i$.
8. A topological space is called *arc connected* if for any two points $a, b \in X$ there exists a continuous path
$$f : [0,1] \to X$$
connecting a and b such that f is a homeomorphism onto its image. Give an example of path connected space that is not arc connected. ($\star$) Sketch an argument showing that path connected open subsets of $\mathbb{R}^n$ are arc connected.
9. Is there is Hausdorff topology τ on $[0,1]$ which is weaker than the usual topology? Weaker here means that every open set in τ is an open set in τ_{Euc}, but there are Euclidean open sets that do not lie in τ.
10. Give an example of a sequence of closed and connected subsets $C_n \subset \mathbb{R}^2$ such that $C_n \supset C_{n+1}$, but the intersection $\bigcap_{n=1}^{\infty}$ is not connected.
11. Let X be a topological space. Define the one-point compactification as follows. Let X^+ be the set underlying X and an additional point ∞. Define a topology on X^+ whose open sets $U \subset X^+$ are of one of the following two forms.
    - U is contained in $X \subset X^+$, and is open.
    - U is obtained as
    $$U = (X \setminus C) \cup \{\infty\},$$
    where C is compact and closed in X.

    Prove that X^+ is a topological space and that it is always compact.
12. Describe the one-point compactification of $\mathbb{R}$ and $\mathbb{R}^2$.

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13. Let $\mathbf{C}[0,1]$ be the set of continuous $\mathbb{R}$-valued functions on $[0,1]$. Equip $\mathbf{C}[0,1]$ with a the supremum metric d_∞, described in the previous example sheet. Prove that the unit ball in $\mathbf{C}[0,1]$ is *not* compact.
14. Let (X,d) be a compact metric space and $f : X \to X$ a continuous map. If f preserves distances, i.e. $d(x,y) = d(f(x), f(y))$ for all $x, y \in X$, show that f is a homeomorphism. Can the compactness hypothesis be dropped?
15. ($\star$) Sketch an argument to show that the loop space of the sphere is connected, but the loop space of the 2-dimensional torus is not.