Example Sheet 1

(1) Show that the sequence $2017, 20017, 200017, \ldots$ converges in the 5-adic topology on $\mathbf{Z}$.

(2) Let $(\mathbf{R}^n, d)$ denote Euclidean n-space. If P, Q, R are points in $\mathbf{R}^n$ such that

$$d(P,Q) + d(Q,R) = d(P,R),$$

show that Q is on the line segment PR. [You may assume that equality holds in the Cauchy–Schwarz inequality $(\sum_{i=1}^n x_i y_i)^2 \leq (\sum_{i=1}^n x_i^2)(\sum_{j=1}^n y_j^2)$ if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are proportional.]

(3) If (X_1, ρ_1), (X_2, ρ_2) are metric spaces, show that we may define a metric ρ on the set $X_1 \times X_2$ by

$$\rho((x_1, x_2), (y_1, y_2)) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2).$$

Show moreover that the projection maps onto the two factors are continuous maps.

Suppose now (X_i, ρ_i) are metric spaces for $i = 1, 2, \ldots$. Let X be the set of all sequences (x_i) with $x_i \in X_i$ for all i; show that we may define a metric $\tilde{\rho}$ on X by

$$\tilde{\rho}((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

(4) Consider the following subsets $A \subset \mathbf{R}^2$, and determine whether they are open, closed or neither.

(a) $A = \{(x,y) \ : \ x < 0\} \cup \{(x,y) \ : \ x > 0, \ y > 1/x\}$;

(b) $A = \{(x, \sin(1/x)) \ : \ x > 0\} \cup \{(0,y) \ : \ -1 \leq y \leq 1\}$;

(c) $A = \{(x,y) \ : \ y \in \mathbf{Q}, \ y = x^n$ for some positive integer $n\}$.

(5) Let $Y = \{0\} \cup \{1/n \ : \ n = 1, 2, \ldots\} \subset \mathbf{R}$ with the standard metric. For (X,d) any metric space, show that the continuous maps $f : Y \to X$ correspond precisely to the convergent sequences $x_n \to x$ in X.

(6) Suppose $F \subset X$ is a subset of a metric space (X, ρ); define a distance function by $\rho(x, F) = \inf_{y \in F} d(x,y)$ and show that it is continuous in x. Show that F is closed if and only if $\rho(x,F) > 0$ for all $x \notin F$. Given disjoint closed sets F_1, F_2 in X, prove that there exist open subsets U_1, U_2 of X with $U_1 \cap U_2 = \emptyset$, $F_1 \subset U_1$ and $F_2 \subset U_2$.

(7) Show that the interior of a (non-degenerate) convex polygon in $\mathbf{R}^2$ is homeomorphic to the open unit disc in $\mathbf{R}^2$, which in turn is homeomorphic to the Euclidean plane $\mathbf{R}^2$, and that a closed convex polygon is homeomorphic to the closed unit disc. Given the fact that a continuous function on a closed rectangle is bounded (proved by 'lion hunting', i.e. keep subdividing rectangles into four smaller ones), determine whether the closed unit disc is homeomorphic to the open unit disc.

(8) Let d_1, d_2, d_∞ be the metrics on $\mathbf{R}^n$ given by $d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$, $d_2(\mathbf{x}, \mathbf{y}) = [\sum_{i=1}^n (x_i - y_i)^2]^{1/2}$ and $d_\infty(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i|$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, show that

$$d_1(\mathbf{x}, \mathbf{y}) \geq d_2(\mathbf{x}, \mathbf{y}) \geq d_\infty(\mathbf{x}, \mathbf{y}) \geq d_2(\mathbf{x}, \mathbf{y})/\sqrt{n} \geq d_1(\mathbf{x}, \mathbf{y})/n.$$

Deduce that the metrics are topologically equivalent (i.e. give rise to the same metric topology on $\mathbf{R}^n$).

(9) Let d_1, d_2, d_∞ be the metrics on $C[0, 1]$ given by $d_1(f, g) = \int_0^1 |f - g|$, $d_2(f, g) = [\int_0^1 (f - g)^2]^{1/2}$ and $d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Show that the corresponding metric topologies on $C[0, 1]$ are distinct.

(10) Let A be a subset of a topological space; show that $\mathrm{Cl}(A)$ is just the set of accumulation points for A. Show that a map $f : X \to Y$ between topological spaces is continuous if and only if $f(\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$ for all subsets $A \subset X$.

(11) Show that the standard metric topology on $\mathbf{R}^n$ has a countable base of open sets. Give an example of a metric topology on $\mathbf{R}^n$ for which this is not true.

(12) Let $f, g : X \to Y$ be two continuous maps, where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \ : \ f(x) = g(x)\}$ is a closed subspace of X. Deduce that the set of fixed points of a continuous map of a Hausdorff topological space to itself is a closed subset.

(13) Let $\mathbf{T} = \{z \in \mathbf{C} \ : \ |z| = 1\}$ be the unit circle, with subspace topology induced from the usual topology on $\mathbf{C}$. We define an equivalence relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if $x - y \in \mathbf{Z}$. Prove that $\mathbf{T}$ is homeomorphic to $\mathbf{R}/\sim$ with the quotient topology.

(14) Suppose that $(X_i, \rho_i) = (\mathbf{R}, d)$ for $i = 1, 2, \ldots$, where d denotes the Euclidean metric, and that $\tilde{\rho}$ denotes the metric defined in Question 3 on the set X of real sequences. Let $Y \subset X$ be the subset of sequences (x_n) with $x_n = 0$ for $n \gg 0$. Show that
(a) we may define a metric ρ' on Y by $\rho'((x_n), (y_n)) = \sum_{n=1}^\infty d(x_n, y_n)$, and
(b) the subspace topology on Y (induced from the $\tilde{\rho}$-metric toplogy on X) is different from the ρ'-metric topology on Y.

(15) Describe all convergent sequences (x_n) for $\mathbf{R}^2$ equipped with the 'British Rail metric' (as described in lectures).

(16) Let A be a subset of a topological space (X, τ). Prove that

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}\ A))) = \mathrm{Cl}(\mathrm{Int}\ A)).$$

*Find a subset $A \subset \mathbf{R}$ for which the operations of taking successive interiors and closures yield precisely seven distinct sets (including A itself).

(17) Consider the two dimensional torus $X = \mathbf{R}^2/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2$ and $y_1 - y_2$ are both integers. Show that X is a metric space by giving an explicit description of a metric inducing the quotient topology. Let $L \subset \mathbf{R}^2$ be the line $y = \alpha x$ for some $\alpha \in \mathbf{R}$; show that there is a continuous map $\phi : L \to X$, and determine when the image of ϕ is a closed subset of X.

(18)* Suppose $p \neq 2$ is prime number. Choose $a \in \mathbf{Z}$ which is not a square and not divisible by p. Suppose $x^2 \equiv a \pmod p$ has a solution x_0. Show that there exists x_1 such that $x_1 \equiv x_0 \pmod p$ and $x_1^2 \equiv a \pmod{p^2}$, and iteratively that there is an x_n such that $x_n \equiv x_{n-1} \pmod{p^n}$ and $x_n^2 \equiv a \pmod{p^{n+1}}$. Show that (x_n) is a Cauchy sequence in $(\mathbf{Q}, d_p)$, where d_p denotes the p-adic metric on $\mathbf{Q}$, and deduce that $(\mathbf{Q}, d_p)$ is not complete.

(19)* Let A be an uncountable set and $X = \{0,1\}^A := \{f : A \to \{0,1\}\}$. For B a countable subset of A and $g : B \to \{0,1\}$, let

$$U_{B,g} := \{f : A \to \{0,1\} \ : \ f(\alpha) = g(\alpha) \text{ for } \alpha \in B\}.$$

Show that the collection of all such subsets of X form a base for a topology on X. Let

$$Y := \{f : A \to \{0,1\} \ : \ f(\alpha) = 0 \text{ for all but countably many } \alpha \in A\} \subset X.$$

For any sequence $(g_n) \in Y$ such that $g_n \to g \in X$, show that $g \in Y$. Show however that Y is dense in X, and so in particular Y is not closed.