Part 1B Metric and Topological Spaces 2012 pmhw@dpmms.cam.ac.uk

Example Sheet 1

(1) Show that the sequence $2012, 20012, 200012, \ldots$ converges in the 5-adic topology on $\mathbf{Z}$.

(2) Let $(\mathbf{R}^n, d)$ denote Euclidean n-space. If P, Q, R are points in $\mathbf{R}^n$ such that

$$d(P, Q) + d(Q, R) = d(P, R),$$

show that Q is on the line segment PR. [You may assume that equality holds in the Cauchy–Schwarz inequality $(\sum_{i=1}^n x_i y_i)^2 \leq (\sum_{i=1}^n x_i^2)(\sum_{j=1}^n y_j^2)$ if and only if the vectors $\mathbf{x}$ and $\mathbf{y}$ are proportional.]

(3) If (X_1, ρ_1), (X_2, ρ_2) are metric spaces, show that we may define a metric ρ on the set $X_1 \times X_2$ by

$$\rho((x_1, x_2), (y_1, y_2)) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2).$$

Show moreover that the projection maps onto the two factors are continuous maps.

Suppose now (X_i, ρ_i) are metric spaces for $i = 1, 2, \ldots$. Let X be the set of all sequences (x_i) with $x_i \in X_i$ for all i; show that we may define a metric $\tilde{\rho}$ on X by

$$\tilde{\rho}((x_n), (y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

(4) Consider the following subsets $A \subset \mathbf{R}^2$, and determine whether they are open, closed or neither.

(a) $A = \{(x, y) \ : \ x < 0\} \cup \{(x, y) \ : \ x > 0,\ y > 1/x\}$;

(b) $A = \{(x, \sin(1/x)) \ : \ x > 0\} \cup \{(0, y) \ : \ -1 \leq y \leq 1\}$;

(c) $A = \{(x, y) \ : \ y \in \mathbf{Q},\ y = x^n$ for some positive integer $n\}$.

(5) Let $Y = \{0\} \cup \{1/n \ : \ n = 1, 2, \ldots\} \subset \mathbf{R}$ with the standard metric. For (X, d) any metric space, show that the continuous maps $f : Y \to X$ correspond precisely to the convergent sequences $x_n \to x$ in X.

(6) Suppose $F \subset X$ is a subset of a metric space (X, ρ); define a distance function $\rho(x, F)$ and show that it is continuous in x. Show that F is closed if and only if $\rho(x, F) > 0$ for all $x \notin F$. Given disjoint closed sets F_1, F_2 in X, prove that there exist open subsets U_1, U_2 of X with $U_1 \cap U_2 = \emptyset$, $F_1 \subset U_1$ and $F_2 \subset U_2$.

(7) Describe all convergent sequences (x_n) for $\mathbf{R}^2$ equipped with the 'British Rail metric' (as described in lectures).

(8) Show that the interior of a (non-degenerate) convex polygon in $\mathbf{R}^2$ is homeomorphic to the open unit disc in $\mathbf{R}^2$, which in turn is homeomorphic to the Euclidean plane $\mathbf{R}^2$. *Is the statement still true if we omit the condition convex?

(9) Let d_1, d_2, d_∞ be the metrics on $\mathbf{R}^n$ given by $d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |x_i - y_i|$, $d_2(\mathbf{x}, \mathbf{y}) = [\sum_{i=1}^n (x_i - y_i)^2]^{1/2}$ and $d_\infty(\mathbf{x}, \mathbf{y}) = \sup_i |x_i - y_i|$. For any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, show that

$$d_1(\mathbf{x}, \mathbf{y}) \geq d_2(\mathbf{x}, \mathbf{y}) \geq d_\infty(\mathbf{x}, \mathbf{y}) \geq d_2(\mathbf{x}, \mathbf{y})/\sqrt{n} \geq d_1(\mathbf{x}, \mathbf{y})/n.$$

Deduce that the metrics are topologically equivalent (i.e. give rise to the same metric topology on $\mathbf{R}^n$).

(10) Let d_1, d_2, d_∞ be the metrics on $C[0,1]$ given by $d_1(f, g) = \int_0^1 |f - g|$, $d_2(f, g) = [\int_0^1 (f - g)^2]^{1/2}$ and $d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Show that the corresponding metric topologies on $C[0,1]$ are distinct.

(11) Let A be a subset of a topological space (X, τ). Prove that

$$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}\, A))) = \mathrm{Cl}(\mathrm{Int}\, A)).$$

*Find a subset $A \subset \mathbf{R}$ for which the operations of taking successive interiors and closures yield precisely seven distinct sets (including A itself).

(12) Let A be a subset of a topological space; show that $\mathrm{Cl}(A)$ is just the set of accumulation points for A.

(13) Show that the standard metric topology on $\mathbf{R}^n$ has a countable base of open sets. Give an example of a metric topology on $\mathbf{R}^n$ for which this is not true.

(14) Let $f, g : X \to Y$ be two continuous maps, where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \ : \ f(x) = g(x)\}$ is a closed subspace of X. Deduce that the set of fixed points of a continuous map of a Hausdorff topological space to itself is a closed subset.

(15) Let $\mathbf{T} = \{z \in \mathbf{C} \ : \ |z| = 1|\}$ be the unit circle, with subspace topology induced from the usual topology on $\mathbf{C}$. We define an equivalence relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if $x - y \in \mathbf{Z}$. Prove that $\mathbf{T}$ is homeomorphic to $\mathbf{R}/\sim$ with the quotient topology.

(16) Suppose that $(X_i, \rho_i)) = (\mathbf{R}, d)$ for $i = 1, 2, \ldots$, where d denotes the Euclidean metric, and that $\tilde{\rho}$ denotes the metric defined in Question 3 on the set X of real sequences. Let $Y \subset X$ be the subset of sequences (x_n) with $x_n = 0$ for $n \gg 0$. Show that
(a) we may define a metric ρ' on Y by $\rho'((x_n), (y_n)) = \sum_{n=1}^{\infty} d(x_n, y_n)$, and
(b) the subspace topology on Y (induced from the $\tilde{\rho}$-metric toplogy on X) is different from the ρ'-metric topology on Y.

(17) Let $P_1, \ldots, P_N$ be distinct points in $\mathbf{R}^2$ with $A = \{P_1, \ldots, P_N\}$ and $X = \mathbf{R}^2/A$, the space where the whole set A is identified to a point. Show that X is a metric space by giving an explicit description of a metric which induces the quotient topology (the usual choice is sometimes known as the 'London Underground metric').

(18) Consider the two dimensional torus $X = \mathbf{R}^2/\sim$, where $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2$ and $y_1 - y_2$ are both integers. Show that X is a metric space, by giving an explicit description of a metric inducing the quotient topology. Let $L \subset \mathbf{R}^2$ be the line $y = \alpha x$ for some $\alpha \in \mathbf{R}$; show that there is a continuous map $\phi : L \to X$, and determine when the image of ϕ is a closed subset of X.

(19)* Let A be an uncountable set and $X = \{0,1\}^A := \{f : A \to \{0,1\}\}$. For B a countable subset of A and $g : B \to \{0,1\}$, let

$$U_{B,g} := \{f : A \to \{0,1\} \ : \ f(\alpha) = g(\alpha) \text{ for } \alpha \in B\}.$$

Show that the collection of all such subsets of X form a base for a topology on X. Let

$$Y := \{f : A \to \{0,1\} \ : \ f(\alpha) = 0 \text{ for all but countably many } \alpha \in A\} \subset X.$$

For any sequence $(g_n) \in Y$ such that $g_n \to g \in X$, show that $g \in Y$. Show however that Y is dense in X, and so in particular Y is not closed.

(20)* Suppose $p \neq 2$ is prime number. Choose $a \in \mathbf{Z}$ which is not a square and not divisible by p. Suppose $x^2 \equiv a \pmod{p}$ has a solution x_0. Show that there exists x_1 such that $x_1 \equiv x_0 \pmod{p}$ and $x_1^2 \equiv a \pmod{p^2}$, and iteratively that there is an x_n such that $x_n \equiv x_{n-1} \pmod{p^n}$ and $x_n^2 \equiv a \pmod{p^{n+1}}$. Show that (x_n) is a Cauchy sequence in $(\mathbf{Q}, d_p)$, where d_p denotes the p-adic metric on $\mathbf{Q}$, and deduce that $(\mathbf{Q}, d_p)$ is not complete.