Metric & Topological Spaces, sheet 2: 2010

1. Which of the following subspaces of $\mathbb{R}^2$ are (a) connected (b) path-connected? $B_{(x,y)}(t)$ denotes the open t-disc about $(x,y) \in \mathbb{R}^2$ and $\overline{X} = cl(X)$ denotes closure.
(i) $B_{(1,0)}(1) \cup B_{(-1,0)}(1)$;
(ii) $\overline{B_{(1,0)}(1)} \cup \overline{B_{(-1,0)}(1)}$
(iii) $B_{(1,0)}(1) \cup \overline{B_{(-1,0)}(1)}$
(iv) $\{(x,y) \,|\, x = 0 \text{ or } y/x \in \mathbb{Q}\}$.

2. (a) Let $\phi : [0,1] \to [0,1]$ be continuous. Prove that ϕ has a fixed point.

   (b) Prove that an odd degree real polynomial has a real root.

   (c) Let $\mathbb{S}^1 \subset \mathbb{R}^2$ denote the unit circle in the Euclidean plane (with the subspace topology) and let $f : \mathbb{S}^1 \to \mathbb{R}$ be continuous. Prove there is some $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.

   (d) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and has $f(0) = f(1)$. For each integer $n \geq 2$ show there is some x s.t. $f(x) = f(x + \frac{1}{n})$.

3. Prove there is no continuous function $f : [0,1] \to \mathbb{R}$ such that $x \in \mathbb{Q} \Leftrightarrow f(x) \notin \mathbb{Q}$ (where $\mathbb{Q}$ denotes the rational numbers).

4. (a) Suppose $A \subset \mathbb{R}^n$ is not compact. Show there is a continuous function on A which is not bounded.

   (b) I am in an infinite forest and can't see out. A troop of renegade beavers gnaw down all but finitely many trees. Can I see out now ?

5. (i) Give an example of a sequence of closed connected subsets $C_n \subset \mathbb{R}^2$ s.t. $C_n \supset C_{n+1}$ but $\bigcap_{n=1}^{\infty} C_n$ not connected.

   (ii) If $C_n \subset X$ is compact and connected in a Hausdorff space, and $C_n \supset C_{n+1}$ for each n, show $\bigcap_{n=1}^{\infty} C_n$ *is* connected.

6. Let X be a topological space. The *one-point compactification* X^+ of X is set-wise the union of X and an additional point ∞ (thought of as "at infinity") with the topology: $U \subset X^+$ is open if either

   (i) $U \subset X$ is open in X or

   (ii) $U = V \cup \{\infty\}$ where $V \subset X$ and $X \backslash V$ is both compact and closed in X.

   Prove that X^+ is a topological space and prove that it is compact (N.B. regardless of whether X is compact or not!).

7. A family of sets has the *finite intersection property* if and only if every *finite* subfamily has non-empty intersection. Prove that a space X is compact if and only if whenever $\{V_a\}_{a \in A}$ is a family of closed subsets of X with the finite intersection property, the whole family has non-empty intersection.

8. Let (X,d) be a compact metric space. Prove that a subspace $Z \subset X$ is compact only if every sequence in Z has a subsequence which converges in the metric to a point of Z. [Note: the question requires "only if" and not "if".]

   Let X be the space of continuous functions from $[0,1]$ to the reals $\mathbb{R}$ with the metric $d(f,g) = sup\{|f(x) - g(x)| \ : \ x \in [0,1]\}$. Prove that the unit ball $\{u \in X \mid d(0,u) \leq 1\}$ is *not compact*, where 0 denotes the obvious zero-function. [Thus the "Heine-Borel" theorem is *not* valid in arbitrary metric spaces.]

9. Let M be a compact metric space and suppose that for every $n \in \mathbb{Z}_{\geq 0}$, $V_n \subset M$ is a closed subset and $V_{n+1} \subset V_n$. Prove that

$$\text{diameter}(\bigcap_{n=1}^{\infty} V_n) \;=\; \inf\{\text{diameter}(V_n) \mid n \in \mathbb{Z}_{\geq 0}\}.$$

[Hint: suppose the LHS is smaller by some amount ϵ.]

10. Fix a prime p and let $a \in \mathbb{Q}$ be non-zero. One can uniquely write $a = p^n \frac{x}{y}$ with x and y coprime, $n \in \mathbb{Z}$ and xy not divisible by p. Define

$$v_p(a) = n; \quad v_p(0) = \infty; \quad \text{and} \quad |a|_p = p^{-v_p(a)},\ |0|_p = 0.$$

(a) Prove $v(a-b) \geq min\{v(a), v(b)\}$ for any $a, b \in \mathbb{Q}$.

(b) Defining $d_p(a,b) = |a-b|_p$, prove that d_p is a metric on $\mathbb{Q}$; this is called the *p-adic* metric, of much importance in number theory.

(c) Show that if p and q are distinct primes, d_p and d_q are inequivalent metrics.

(d)* Suppose $p \neq 2$. Choose $a \in \mathbb{Z}$ which is not a square in $\mathbb{Q}$ and which is not divisible by p. Suppose $x^2 \equiv a \pmod p$ has a solution. Show there is x_1 s.t. $x_1 \equiv x_0 \pmod p$ and $x_1^2 \equiv a \bmod p^2$, and iteratively that there is x_n s.t. $x_n \equiv x_{n-1} \bmod p$ and $x_n^2 \equiv a \bmod p^{n+1}$. Show that (x_n) is a Cauchy sequence in $(\mathbb{Q}, d_p)$ with no convergent subsequence. Deduce $(\mathbb{Q}, d_p)$ is not complete.

[Any incomplete metric space admits a canonical *completion*. The completion of $(\mathbb{Q}, d_{eucl})$ is $\mathbb{R}$, a general element of which can be written as $a = \sum_{k=m}^{\infty} a_k 10^{-k}$ with $0 \leq a_k \leq 9$. The completion $\mathbb{Q}_p$ of $(\mathbb{Q}, d_p)$ comprises the expressions $\sum_{i=m}^{\infty} a_i p^i$ with $0 \leq a_i \leq p-1$.]

11. * (a) Draw an example of a smooth connected surface in $\mathbb{R}^3$ with infinitely many "ends" (i.e. for which complements of arbitrarily large compact sets have infinitely many connected components). Hence, or otherwise, draw three pairwise non-homeomorphic connected infinite genus smooth surfaces in $\mathbb{R}^3$. (The *genus* is the number of holes: muffins have genus 0, bagels have genus 1, pretzels have genus 3.)

(b) Sketch an informal argument to explain why the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are pairwise not homeomorphic.

(c) Let X be a topological space and $x_0 \in X$ a distinguished point. Show that the set of connected components of the based loop space $\Omega X = \{\gamma : [0,1] \to X \mid \gamma \text{ is continuous, } \gamma(0) = \gamma(1) = x_0\}$ forms a group.

(d) Give examples in which this group is non-trivial. Can it be non-trivial and finite? Can it be non-abelian?

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