Metric & Topological spaces, Sheet 1: 2010

1. (a) Give a bounded open subset of $(\mathbb{R}, \mathcal{T}_{eucl})$ which is a not a finite union of open intervals.

   (b) Is the trigonometric function $sin : (\mathbb{R}, \mathcal{T}_{Zariski}) \to (\mathbb{R}, \mathcal{T}_{Zariski})$ continuous?

   (c) Give a topology on $\mathbb{R}$ which is neither trivial nor discrete such that every open set is closed (and vice-versa).

   (d) Let $\mathcal{T}$ be the topology on $\mathbb{R}$ for which open sets are ϕ, $\mathbb{R}$ and open intervals of the form $(-\infty, a)$. Show this is a topology, and describe the closure of the singleton set $\{a\}$. What are the continuous functions from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T}_{eucl})$?

   (e) Give $\mathbb{R}$ the following topology: a subset H is closed in $\mathbb{R}$ if and only if H is closed and bounded in the usual (Euclidean metric) topology. Show that this *is* a topology, that points are closed sets, but that this topology is not Hausdorff.

   (f) Define a subset of the integers $\mathbb{Z}$ to be open either if it is empty or if for some $k \in \mathbb{Z}$ the set S contains all integers $\geq k$. Show this defines a topology. Is it metrisable ?

2. Which of the following are open in $(\mathbb{R}^2, \mathcal{T}_{eucl})$? [Convincing pictures are sufficient.]

   (i) $\{y > x^2\}$

   (ii) $\{y > x^2, y \leq 1\}$

   (iii) $\{y > x^2, y \leq -1\}$.

3. Prove or give counterexamples to:

   (i) A continuous function $f : X \to Y$ is an *open map* i.e. if $U \subset X$ is an open subset then $f(U)$ is an open subset of Y.

   (ii) If $f : X \to Y$ is continuous and bijective (that is, one-to-one and onto) then f is a homeomorphism.

   (iii) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

4. Let X and Y be topological spaces.

   (a) If $X = A \cup B$ is a union of (not necessarily disjoint) closed subsets, prove that a function f on X is continuous if and only if $f|_A$ and $f|_B$ are continuous functions on A, B respectively, where A, B have the subspace topology induced from X. (The notation refers to the restriction of the function to the appropriate domain.)

   (b) Show $f : X \to Y$ is continuous if and only if for all $A \subset X$, $f(cl(A)) \subset cl(f(A))$. Deduce that if f is surjective, the continuous image of a dense set is dense.

   (c) If now X and Y are metric spaces, show $f : X \to Y$ is continuous if it preserves limits of sequences, i.e. if for every sequence $(x_n) \subset X$ converging to $a \in M_1$, the sequence $(f(x_n)) \subset Y$ converges to $f(a)$.

5. Let $f, g : X \to Y$ be continuous functions where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \mid f(x) = g(x)\}$ is a closed subspace of X. Deduce that the fixed point set of a continuous function on a Hausdorff space is closed.

6. Let $X = \mathbb{Z}_{>0}$ be the set of strictly positive integers. Define a topology on X by saying that the *basic* open sets are the arithmetic progressions

$$U_{a,b} = \{na + b \in X \mid n \in \mathbb{Z}\}$$

for pairs (a,b) with $hcf(a,b)=1$. (Hence, a general open set is a union of these.)

(a) Show this does define a topology. Is it Hausdorff ?

(b) If p is prime, show that $U_{p,0}$ is a closed subset of X. Deduce there are infinitely many prime numbers.

7. (a) Show the quotient space $([0,1]\cup[2,3])/1\sim 2$ is homeomorphic to a closed interval.

   (b) Define an equivalence relation $\sim$ on the interval $[0,1]\subset\mathbb{R}$ by $x\sim y \iff x-y\in\mathbb{Q}$. Describe the quotient space $I/\sim$.

8. Let S_1 be the quotient space given by identifying the north and south poles on the 2-sphere. Let S_2 be the quotient space given by collapsing one circle $S^1\times\{pt\}$ inside the 2-dimensional torus $S^1\times S^1$ to a point. Draw pictures of S_1 and S_2 and sketch an argument to show that they are homeomorphic.

9. (a) Prove that the product of two metric spaces admits a metric inducing the product topology.

   (b) Show that if the product of two metric spaces is complete, then so are the factors.

   (c) Let M denote the space of bounded sequences of real numbers with the *sup* metric. Show (i) the subspace of convergent sequences is complete and (ii) the subspace of sequences with only finitely many non-zero values is not complete.

   (d) Let M be a complete metric space and $f: M\to M$ be continuous. Suppose for some r the iterate $f^{\circ r}=f\circ\cdots\circ f$ (r times) is a contraction. Prove f has a unique fixed point.

10. Let G be a topological group; so G is a topological space and there are given a distinguished point $e\in G$, continuous functions $m: G\times G\to G$ and $i: G\to G$ (multiplication and inverse) which satisfy the (usual, algebraic) group axioms. Typical examples are matrix groups like $SL_2(\mathbb{R})$, $SO(3)$, the unit circle in $\mathbb{C}$, etc. Prove the following:

    (a) G is homogeneous: given any $x,y\in G$ there is a homeomorphism $\phi: G\to G$ such that $\phi(x)=y$.

    (b) If $\{e\}\subset G$ is a closed subset, then the diagonal $\Delta G=\{(g,g)\mid g\in G\}\subset G\times G$ is a closed subgroup of $G\times G$.

    (c) If $\{e\}$ is closed in G then the centre $Z(G)=\{g\in G\mid gh=hg\ \forall h\in G\}$ is a closed normal subgroup of G.

    (d) Let H be an algebraic subgroup of G. Give the set of cosets $(G:H)$ the quotient topology from the natural projection map $\pi: G\to(G:H)$. Prove that π is an open map (images of open sets are open).

    (e) Prove $(G:H)$ is Hausdorff if and only if H is closed in G.

11. (a) Show that a space X may be homeomorphic to a subspace of a space Y whilst Y is homeomorphic to a subspace of X, but where X,Y are not themselves homeomorphic.

    (b)* Give an example of a pair of spaces X,Y which are not homeomorphic but for which $X\times I$ and $Y\times I$ are homeomorphic, where I denotes the unit interval with its usual topology. [Hint: draw a flat handbag, i.e. two arcs attached to a disk.]

Ivan Smith
is200@cam.ac.uk