Example Sheet 2

0. Suppose that $X_1 \times X_2$ are topological spaces and that $t \in X_1$. Show that $\{t\} \times X_2$, considered with the subspace topology induced from the product topology on $X_1 \times X_2$, is homeomorphic to X_2.

1. Is the space $C[0,1]$, with the topology induced by the max metric, a connected topological space?

2. Let $A \subseteq \mathbb{R}^2$ be the set of all points with at least one rational coordinate. Is A connected? What if the points with *both* coordinates rational are removed from A?

3. Is there an infinite compact subset of $\mathbb{Q}$?

4. Show that there is no continuous injective map from $\mathbb{R}^2$ to $\mathbb{R}$ [*Hint: consider the induced map on* $\mathbb{R}^2 \setminus \{0\}$].

5. Define the *Riemann sphere* $\mathbb{C}_\infty$ to be the complex plane $\mathbb{C}$ together with an extra point called ∞, and with the following topology. A basis for the open sets in $\mathbb{C}_\infty$ consists of the usual open sets in $\mathbb{C}$ together with the sets of the form $\{\infty\} \cup \{z : |z| > r\}$. Show that $\mathbb{C}_\infty$ is a compact topological space containing a homeomorphic copy of $\mathbb{C}$.

6. Which of the following topological spaces are compact: $C[0,1]$, $\mathbb{R}$ with the cocountable topology, the Klein bottle?

7. Suppose that $X = \{0,1\}^{\mathbb{N}}$ is endowed with the metric

$$d((x_i),(y_i)) = \sum_i 2^{-i}|x_i - y_i|.$$

Show directly that X is sequentially compact. [*You may assume any version of the axiom of choice.*]

8. Show that X is connected if and only if the only continuous functions $f : X \to \mathbb{Z}$ are the constant functions. Is the same true if $\mathbb{Z}$ is replaced by $\mathbb{Q}$?

9. Suppose that X is connected and that $f : X \to \mathbb{R}$ is *locally constant*, that is to say for every $x \in X$ there is an open set U containing x on which f is constant. Show that f is constant.

10. Are the rationals with the 2-adic topology (that is, the topology induced by the 2-adic metric) connected?

11. I am stood in the middle of a forest on $\mathbb{R}^2$ and cannot see anything but trees in every direction. Is it necessarily possible to remove all but finitely many trees so that I still can't see out?

12. Is the following statement true: for every compact metric space X there is a constant N such that every subcover of X by balls of radius one has a subcover with at most N balls?

13. Is there a metric on $\mathbb{N}$ which makes it into a connected topological space?

14. What are the connected components of the second space in Q2?

15. Let C_n, $n \in \mathbb{N}$, be compact, connected, nonempty subsets of a Hausdorff space X such that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$. Prove that the intersection $\bigcap_{n\in\mathbb{N}} C_n$ is connected. Show by example that the compactness assumption may not be dropped.

16. Show that there is a constant K such that for any $x, y \in \mathbb{R}$ satisfying $x^2 + y^2 = 2$ we have $(x-y)^2 \leqslant K\big(4 - (x+y)^2\big)$.

b.j.green@dpmms.cam.ac.uk