Metric & Topological Spaces, sheet 2: (2006)

1. Which of the following subspaces of $\mathbb{R}^2$ are (a) connected (b) path-connected? $B_t(x,y)$ denotes the open t-disc about $(x,y) \in \mathbb{R}^2$ and $\overline{X} = cl(X)$ denotes closure.
   (i) $B_1(1,0) \cup B_1(-1,0)$;
   (ii) $\overline{B_1(1,0)} \cup \overline{B_1(-1,0)}$
   (iii) $B_1(1,0) \cup \overline{B_1(-1,0)}$
   (iv) $\{(x,y) \mid x = 0 \text{ or } y/x \in \mathbb{Q}\}$.

2. Show the product of two connected spaces is connected.

3. Is $\mathbb{C}$, with the Zariski topology (open sets are complements of finite sets, and the empty set), connected?

4. Prove there is no continuous injection $\mathbb{R}^2 \to \mathbb{R}$.

5. Let $Z = \{0,1\}$ denote the two-point set with the discrete topology. Prove that a topological space X is connected if and only if every continuous function $\lambda : X \to Z$ is constant. Deduce that every path-connected space is connected.

   It is a fact that a connected open subset U of the plane $\mathbb{R}^2$ is path-connected. Is this true without the assumption that U be open?

6. Let X be a topological space and $H \subset X$ a connected subset. Prove that $cl(H)$ is connected.

7. (a) Let $\phi : [0,1] \to [0,1]$ be continuous. Prove (using connectedness) that ϕ has a fixed point.

   (b) Let $\mathbb{S}^1 \subset \mathbb{R}^2$ denote the unit circle in the Euclidean plane (with the subspace topology) and let $f : \mathbb{S}^1 \to \mathbb{R}$ be continuous. Prove there is some $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.

8. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and has $f(0) = f(1)$. For each integer $n \geq 2$ show there is some x s.t. $f(x) = f(x + \frac{1}{n})$.

9. (i) Give an example of a sequence of closed connected subsets $C_n \subset \mathbb{R}^2$ s.t. $C_n \supset C_{n+1}$ but $\bigcap_{n=1}^{\infty} C_n$ not connected.

   (ii) If $C_n \subset X$ is compact and connected in a Hausdorff space, and $C_n \supset C_{n+1}$ for each n, show $\bigcap_{n=1}^{\infty} C_n$ *is* connected.

10. Suppose $A \subset \mathbb{R}^n$ is not compact. Show there is a continuous function on A which is not bounded.

11. A continuous function $f : X \to Y$ is *proper* if $f(C) \subset Y$ is closed whenever $C \subset X$ is closed and the preimages of points in Y are compact subsets of X. Prove a continuous function f from a compact space X to a Hausdorff space Y is proper. Deduce that if f is also a bijection then X and Y are homeomorphic.

12. Let X be a topological space. The *one-point compactification* X^+ of X is set-wise the union of X and an additional point ∞ (thought of as "at infinity") with the topology: $U \subset X^+$ is open if either

    (i) $U \subset X$ is open in X or

    (ii) $U = V \cup \{\infty\}$ where $V \subset X$ and $X \backslash V$ is both compact and closed in X.

    Prove that X^+ is a topological space and prove that it is compact (N.B. regardless of whether X is compact or not!).

13. Using connectedness arguments, prove that the three intervals $[0,1], (0,1)$ and $[0,1)$ are pairwise not homeomorphic. Sketch a proof of the same result based on compactness arguments.

14. Prove that a discrete space is totally disconnected. Does the converse always hold? Can you give an example of a compact totally disconnected subset of $\mathbb{R}$?

15. A family of sets has the *finite intersection property* if and only if every *finite* subfamily has non-empty intersection. Prove that a space X is compact if and only if whenever $\{V_a\}_{a \in A}$ is a family of closed subsets of X with the finite intersection property, the whole family has non-empty intersection.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd degree polynomial function. Prove that f has a real root.

17. Prove there is no continuous function $f : [0,1] \to \mathbb{R}$ such that $x \in \mathbb{Q} \Leftrightarrow f(x) \notin \mathbb{Q}$ (where $\mathbb{Q}$ denotes the rational numbers).

18. A continuous surjective map $p : E \to B$ is a *covering* if every point $b \in B$ has an open neighbourhood V such that the connected components of $p^{-1}(V)$ are open in E, and p restricts to a homeomorphism from each such component onto V. Prove in this case that a map f from the unit interval $[0,1]$ to B may be "lifted" to E: there is some $g : [0,1] \to E$ such that $p \circ g = f$. How unique is g?

19. Prove that the map $\mathbb{R} \to \mathbb{S}^1$ taking a real number t to the unit complex number e^{it} is a covering map.

20. Let $\mathbb{R}^n$ denote Euclidean n-dimensional space with its usual metric topology. Prove that $\mathbb{R} = \mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic. Can you prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic?

21. Let (X, d) be a compact metric space. Prove that a subspace $Z \subset X$ is compact only if every sequence in Z has a subsequence which converges in the metric to a point of Z. [Note: the question requires "only if" and not "if".]

    Let X be the space of continuous functions from $[0,1]$ to the reals $\mathbb{R}$ with the metric $d(f,g) = sup\{|f(x) - g(x)| \ : \ x \in [0,1]\}$. Prove that the unit ball $\{u \in X \mid d(0,u) \leq 1\}$ is *not compact*, where 0 denotes the obvious zero-function. [Thus the "Heine-Borel" theorem is *not* valid in arbitrary metric spaces.]

22. Prove directly (i.e. not using any equivalence to other forms of compactness) that (a) a sequentially compact metric space is bounded and (b) the product of two sequentially compact metric spaces is sequentially compact.

23. Let M be a compact metric space and suppose that for every $n \in \mathbb{Z}_{\geq 0}$, $V_n \subset M$ is a closed subset and $V_{n+1} \subset V_n$. Prove that

$$\text{diameter}(\bigcap_{n=1}^{\infty} V_n) \;=\; \inf\{\text{diameter}(V_n) \mid n \in \mathbb{Z}_{\geq 0}\}.$$

[Hint: suppose the LHS is smaller by some amount ϵ. Then for each n there is some pair of points $x_n, y_n \in V_n$ separated by more than LHS+ϵ.]

24. Let X be a compact topological space. Prove that for any topological space T the second projection map $X \times T \to T$ is a closed map (i.e. the image of any closed set is closed).

25. Let $f : X \to Y$ be an arbitrary function, Y be a compact space and suppose the graph $\Gamma_f \subset X \times Y$ is closed. Prove that f is continuous.

26. Prove that if the continuous map ϕ is proper (cf. question 11) then $\phi \times \text{id} : X \times T \to Y \times T$ is closed, for any topological space T and $\text{id} : T \to T$ the identity map.

(Hard) We now prove the converse to q.24: a space X is compact if for all spaces T the second projection $p_T : X \times T \to T$ is closed. Let $\mathcal{U} = \{U_a\}_{a \in A}$ be some open cover of X. Form a new cover by adding all finite unions of sets U_a to $\mathcal{U}$. We will obtain a contradiction assuming that $U \neq X$ for all elements of the new cover.

(a) Let X' be the set comprised of X and an additional point P. Show we can define a topology for X' by taking a basis of open sets those of the form (i) $X' \backslash U_a$ for some U_a in the cover (ii) $W \cap (X \backslash U_a)$ for W open in X and any U_a.

(b) Let Γ denote the graph of the (*not necessarily continuous*) inclusion map from $X \to X'$. Show $P \in \pi(Cl(\Gamma))$ where π denotes the second projection map $X \times X' \to X'$.

(c) From above, we know $(x, P) \in Cl(\Gamma)$ for some $x \in X$. Prove this point x does not belong to any of the sets U of the cover, and hence obtain a contradiction.

(d) Deduce that if X, Y are both compact spaces then $X \times Y$ is also compact.

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